CELLULAR ALGEBRAS ARISING FROM HECKE ALGEBRAS OF TYPE H_n

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Abstract. We study a finite-dimensional quotient of the Hecke algebra of type H_n for general n, using a calculus of diagrams. This provides a basis of monomials in a certain set of generators. Using this, we prove a conjecture of C.K. Fan about the semisimplicity of the quotient algebra. We also discuss the cellular structure of the algebra, with certain restrictions on the ground ring.

Proposed Running Title.

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0. Introduction

There has been much recent interest in the Temperley–Lieb algebra and its various generalisations. Graham [**4**] in his thesis studied a certain quotient, which we will call $TL(X)$, of a Hecke algebra $\mathcal{H}(X)$ associated to a Dynkin diagram X. In the case where X is a Dynkin diagram of type A, this quotient was considered by Jones [**8**], who pointed out that it is nothing other than the Temperley–Lieb algebra, which first appeared in [**12**]. The Temperley–Lieb algebra has applications in several areas of mathematics, including statistical mechanics and knot theory.

A remarkable feature of the algebras $TL(X)$ is that they can be finite dimensional, even when $\mathcal{H}(X)$ is infinite dimensional. Graham [**4**] classified the finite dimensional algebras $TL(X)$ into seven infinite families: A, B, D, E, F, H and I.

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(Contrast this to the classification of Hecke algebras associated to irreducible Coxeter systems, in which there are only finitely many algebras of types E, F and H.)

This paper is concerned with the infinite family of type H, in which case the Hecke algebra $\mathcal{H}(H_n)$ is finite dimensional only for $n \leq 4$. The algebra $TL(H_n)$ was mentioned briefly by Fan in [**1**, §7.3], where it was conjectured that $TL(H_n)$ is generically semisimple. The dimensions of the generically irreducible modules are also conjectured. In the course of the paper, we will prove these conjectures. Note that semisimplicity is obvious in the cases where $\mathcal{H}(X)$ is finite dimensional, because in this case $\mathcal{H}(X)$ is itself generically semisimple, but this argument fails if $\mathcal{H}(X)$ is infinite dimensional.

Our approach is first to realise $TL(H_n)$ as an algebra of diagrams arising from the category of "decorated tangles" which was introduced by the author in [**6**]. Diagram calculi have already been developed for algebras of some of the other types: for $TL(A_n)$ it is well known (see [**13**] or [**5**, §6]), types B_n and D_n were done in [**6**], and the infinite-dimensional "affine" Temperley–Lieb algebra $TL(\widehat{A}_n)$ was tackled in [**2**]. This is interesting to do in its own right, since many natural questions about the algebras (such as the determination of the cells, dimensions and structure constants) have simple formulations in terms of the combinatorics of the associated diagrams. We will show that the diagrams may be adapted into a datum for a cellular algebra (in the sense of [**5**]), provided that the polynomial $x^2 - x - 1$ splits into distinct linear factors over the ground ring. Since the algebra $TL(H_2)$ is a q-analogue of a 9-dimensional quotient of the group algebra of the dihedral group of order 10, it seems that such a hypothesis cannot be usefully weakened.

The algebras $TL(X)$ of types A, B, D, E and F each have a basis consisting of monomials in the obvious set of algebra generators (which correspond to the Coxeter generators), and the structure constants with respect to this basis are positive in a natural sense. Furthermore, the product of two monomials is a scalar multiple of another monomial. In type H, the obvious basis of monomials does not have the positivity property, and it is not true that the product of two monomials is a scalar multiple of one other. This means that Fan's techniques from [**1**] are unsuitable for analysing the algebra $TL(H_n)$.

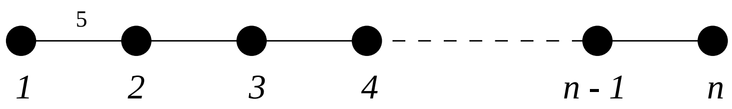
In this paper, we overcome this problem by working with the basis of diagrams, which has much more convenient properties (e.g. positivity of structure constants and compatibility with cellular algebras). This basis is not obvious from the description of $TL(H_n)$ via generators and relations, but is very natural from the viewpoint of decorated tangles. We also show how the new basis elements can be expressed as monomials in a slightly larger set of algebra generators.

1. Preliminaries

1.1 Coxeter groups of type H_n.

Let $n \in \mathbb{N}$ be at least 2. The Coxeter group of type H_n corresponds to the Coxeter graph shown in Figure 1.

Figure 1. Coxeter graph of type H_n



Definition 1.1.1. The Coxeter group $W(H_n)$ is given by generating involutions $\{s_i : i \leq n\}$ and defining relations

$$s_i s_j = s_j s_i \quad \text{if } |i-j| > 1,$$
$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1 \text{ and } \{i,j\} \neq \{1,2\},$$
$$s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2 s_1 s_2.$$

Remark 1.1.2. The group $W(H_2)$ is isomorphic to the dihedral group of order 10 (i.e. $W(I_2)$), but it will be convenient to regard it as a Coxeter group of type H in some of our proofs.

As explained in [**7**, §2], the groups $W(H_n)$ are finite for $n = 2, 3, 4$, where they have orders 10, 120 and 14400 respectively. These groups occur as the full symmetry groups of Platonic solids with pentagonal faces: H_2 corresponds to the pentagon, H_3 to the dodecagon and H_4 to a regular 120-sided solid in 4 dimensions. For $n > 4$, the group $W(H_n)$ is infinite, which is reminiscent of the fact that there is no analogue of the dodecagon in higher dimensions, the only Platonic solids being generalized tetrahedra, cubes and octahedra.

1.2 Hecke algebras of type H_n.

We now introduce the Hecke algebra and its quotient $TL(H_n)$.

Definition 1.2.1. The Hecke algebra $\mathcal{H}(H_n)$ is defined over the ring

$$\mathcal{A} := \mathbb{Z}[v, v^{-1}],$$

where $v = q^{1/2}$. It has a free $\mathcal{A}$-basis $\{T_w : w \in W(H_n)\}$, and the multiplication is defined by the rules

$$T_s T_w := \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{otherwise.} \end{cases}$$

Here, $\ell(w)$ is the length of w, i.e. the length of a shortest word in the s_i which is equal to w.

Following Fan [**1**, §7.3] and Graham [**4**], we make the following definition.

Definition 1.2.2. Let $n \in \mathbb{N} \geq 2$. We define the associative, unital algebra $TL(H_n)$ over $\mathcal{A}$ via generators $E_1, E_2, \dots E_n$ and relations

$$E_i^2 = [2]E_i,$$

$$E_i E_j = E_j E_i \quad \text{if } \ |i-j| > 1,$$

$$E_i E_j E_i = E_i \quad \text{if } \ |i-j| = 1 \ \text{ and } \ i, j > 1,$$

$$E_i E_j E_i E_j E_i = 3E_i E_j E_i - E_i \quad \text{if } \ \{i, j\} = \{1, 2\}.$$

Here, $[2]$ denotes the Laurent polynomial $v + v^{-1}$.

Remark 1.2.3. The algebra $TL(H_n)$ is a quotient of $\mathcal{H}(H_n)$ which corresponds to the Coxeter graph in Figure 1. The quotient map takes the Kazhdan–Lusztig basis element $C'_s = v^{-1}T_e + v^{-1}T_s$, where e is the identity and s is of length 1, to E_s.

Later, we shall want to replace the base ring $\mathcal{A}$ with a field, but we will not be concerned with trying to generalise the results to characteristics 2, 3 or 5.

It is convenient for later purposes to define the following elements of $TL(H_n)$.

Definition 1.2.4. We define

$$\alpha := E_1 E_2 - 1,$$

$$\beta := E_2 E_1 - 1,$$

$$\varepsilon := E_1 E_2 E_1 - 2E_1 \text{ and}$$

$$\zeta := E_2 E_1 E_2 - 2E_2.$$

Remark 1.2.5. Notice that we can rephrase the non-monomial relations in Definition 1.2.2 as the monomial relations $\varepsilon\beta = E_1$ and $\zeta\alpha = E_2$.

2. The diagram algebra Δ_n

We define a calculus of diagrams which will be seen in §3 to describe the generalized Temperley–Lieb algebra of type H. A convenient way to explain this is via the category of "decorated tangles" which was introduced by the author in [**6**].

2.1 The category of decorated tangles.

Following [**3**], we define a tangle as follows.

Definition 2.1.1. A tangle is a portion of a knot diagram contained in a rectangle. The tangle is incident with the boundary of the rectangle only on the north and south faces, where it intersects transversely. The intersections in the north (respectively, south) face are numbered consecutively starting with node number 1 at the western (i.e. the leftmost) end.

Two tangles are equal if there exists an isotopy of the plane carrying one to the other such that the corresponding faces of the rectangle are preserved setwise.

We call the edges of the rectangular frame "faces" to avoid confusion with the "edges" which are the arcs of the tangle.

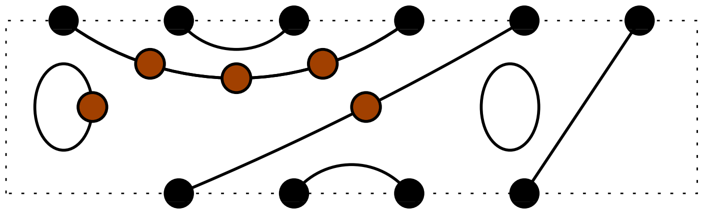
We extend the notion of a tangle so that each arc of the tangle may be assigned a nonnegative integer. (This is similar to the notion of "coloured" tangles in [**3**].) If an arc is assigned the value r, we represent this pictorially by decorating the arc with r blobs. We also require some further restrictions, as explained in the following definition.

Definition 2.1.2. A decorated tangle is a crossing-free tangle in which each arc is assigned a nonnegative integer. Any arc not exposed to the west face of the rectangular frame must be assigned the integer 0.

Remark 2.1.3. This means that any decorated tangle consists only of loops and edges, none of which intersect each other.

Example 2.1.4. Figure 2 shows a typical example of a decorated tangle. We will tend to emphasise the intersections of the tangle with the frame rather than the frame itself, which is why each node (i.e. intersection point with the frame) is denoted by a disc. In this case, the only edges or loops exposed to the west wall are the three which already carry decorations.

FIGURE 2. A decorated tangle



We now define a category based on the set of decorated tangles, as follows.

Definition 2.1.5. The category of decorated tangles, $\mathbb{DT}$, has as its objects the natural numbers (not including zero). The morphisms from n to m are the decorated tangles with n nodes in the north face and m in the south. The source of a morphism is the number of points in the north face of the bounding rectangle, and the target is the number of points in the south face. Composition of morphisms works by concatenation of the tangles, matching the relevant south and north faces together.

Remark 2.1.6. Note that for there to be any morphisms from n to m, it is necessary that $n + m$ be even. Also notice that the asymmetric properties of the west face of the rectangle mean that we cannot introduce the tensor product of two morphisms by the lateral juxtaposition of diagrams as in [**3**].

The category-theoretic definition allows us to define an algebra of decorated tangles, as follows.

Definition 2.1.7. Let R be a commutative ring and let n be a positive integer. Then the R-algebra $\mathbb{DT}_n$ has as a free R-basis the morphisms from n to n, where the multiplication is given by the composition in $\mathbb{DT}$.

Definition 2.1.8. The edges in a tangle T which connect nodes (i.e. not the loops) may be classified into two kinds: propagating edges, which link a node in the north face with a node in the south face, and non-propagating edges, which link two nodes in the north face or two nodes in the south face.

2.2 H-admissible diagrams.

We introduce the concept of an H-admissible diagram, which plays a key rôle in describing the diagram calculus relevant for $TL(H_{n-1})$.

Definition 2.2.1. An H-admissible diagram with n strands is an element of $\mathbb{DT}_n$ with no loops which satisfies the following conditions.

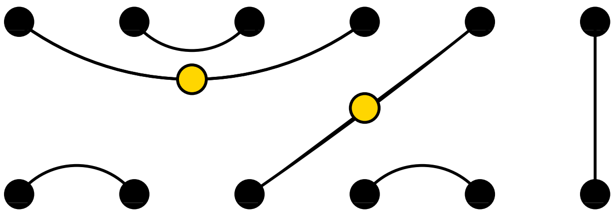
(i) No edge may be decorated if all the edges are propagating.

(ii) If there are non-propagating edges in the diagram, then either there is a decorated edge in the north face connecting nodes 1 and 2, or there is a non-decorated edge in the north face connecting nodes i and $i+1$ for $i > 1$. A similar condition holds for the south face.

(iii) Each edge carries at most one decoration.

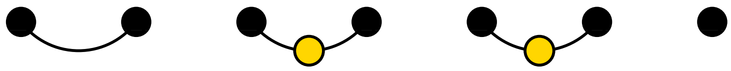
An example of an H-admissible diagram for $n = 6$ is shown in Figure 3.

FIGURE 3. An H-admissible diagram



The point of part (ii) in Definition 2.2.1 excludes situations like the one where Figure 4 appears as the top half of an element of $\mathbb{DT}_n$.

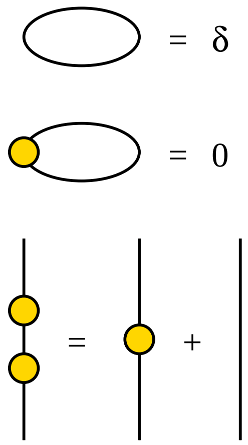
FIGURE 4. The north face of a diagram
excluded by Definition 2.2.1 (ii)



Definition 2.2.2. The algebra Δ_n (over a commutative ring with identity) has as a basis the H-admissible diagrams with n strands and multiplication induced from that of $\mathbb{DT}_n$ subject to the relations shown in Figure 5.

Remark 2.2.3. The meaning of the first relation in Figure 5 is that any undecorated loop can be removed and the resulting tangle multiplied by δ. The second relation means that any tangle containing a loop with one decoration is equivalent to $0 \in \Delta_n$. The third relation means that any tangle T containing an edge or loop ε with r decorations $r > 1$ is equivalent to the sum of two other tangles T' and T'' which

FIGURE 5. Reduction rules for Δ_n



are the same as T except that the edge or loop corresponding to ε carries $r-1$ (respectively, $r-2$) decorations.

The second rule can be modified so that its removal corresponds to multiplication by a second parameter, δ'. This would eventually lead to a two-parameter version of $TL(H_{n-1})$, but we do not pursue this here.

Lemma 2.2.4. *The relations in Figure 5 allow the product of two elements of Δ_n to be expressed unambiguously as a linear combination of basis elements. This makes Δ_n into an associative algebra.*

Proof. We observe that the product of two H-admissible diagrams can be expressed as a linear combination of others by using the reduction rules given.

A case by case check shows that the order in which the relations are applied is immaterial and that the end result can therefore be expressed unambiguously in terms of the basis of H-admissible diagrams.

Using these observations, associativity is inherited from the associativity of $\mathbb{DT}_n$, by consideration of the concatenation of three tangles $TT'T''$. $\square$

3. REALISATION OF $TL(H_n)$ AS AN ALGEBRA OF DIAGRAMS

3.1 Representation of $TL(H_n)$ by diagrams.

One of our main aims will be to show that Δ_{n+1} and $TL(H_n)$ are isomorphic. To do this, we show how to represent $TL(H_n)$ using the H-admissible diagrams.

Definition 3.1.1. The H-admissible diagram U_i, where $1 \leq i \leq n$, is the diagram all of whose edges are propagating and undecorated, except for those attached to nodes i and $i+1$ in the north row, and nodes i and $i+1$ in the south row. These four nodes are connected in the pairs given, using decorated edges if $i = 1$, and using undecorated edges if $i > 1$.

Examples. When $n = 6$, the elements U_1 and U_2 are as shown in Figures 6 and 7.

FIGURE 6. The diagram U_2 for $n = 6$

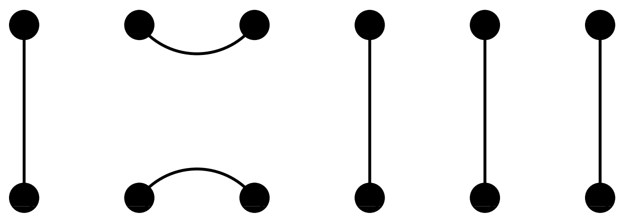
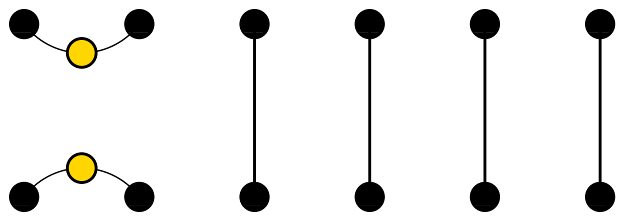


FIGURE 7. The diagram U_1 for $n = 6$



From now on, we take the base ring for Δ_{n+1} to be $\mathcal{A}$, meaning that the parameter δ is $[2]$. More general results may be found by tensoring over a suitable ring.

Proposition 3.1.2. *There is a homomorphism of $\mathcal{A}$-algebras from $TL(H_n)$ to Δ_{n+1} which takes E_i to U_i for each i.*

Proof. This is simply a matter of checking that all the relations in Definition 1.2.2 hold, which presents no problems. □

3.2 Algebra generators for Δ_{n+1}.

In order to prove that ρ is an isomorphism, we will first show that Δ_{n+1} is generated as an $\mathcal{A}$-algebra (with identity) by the elements U_i.

During the course of the proofs, it helps to understand the case $n = 2$, which was the motivation for Definition 1.2.4.

Lemma 3.2.1. *The map ρ is an isomorphism for $n = 2$. The basis of H-admissible diagrams consists of the images of the 9 elements*

$$1, E_1, E_2, E_1E_2, E_2E_1, E_1\beta, E_2\alpha, E_1\zeta, E_2\varepsilon = \zeta E_1.$$

Thus, Δ_3 is generated as an $\mathcal{A}$-algebra with 1 by U_1 and U_2.

Proof. This is another routine exercise using the diagram multiplication, which is instructive to carry out. □

To deal with the case for general n, it is convenient to introduce a number of "moves", in which a diagram element is multiplied (on the left or on the right) by a monomial in the generators

$$G_n := \{1, U_1, \dots, U_n, \rho(\alpha), \rho(\beta), \rho(\zeta)\}$$

to form another diagram element. It will eventually turn out that any H-admissible diagram may be obtained as a suitable word in the generators G_n.

In the next five lemmas, D is an H-admissible diagram. The proofs of the lemmas are all immediate from the diagram multiplication.

Lemma 3.2.2. *Assume D has a propagating edge, E, connecting node p_1 in the north face to node p_2 in the south face.*

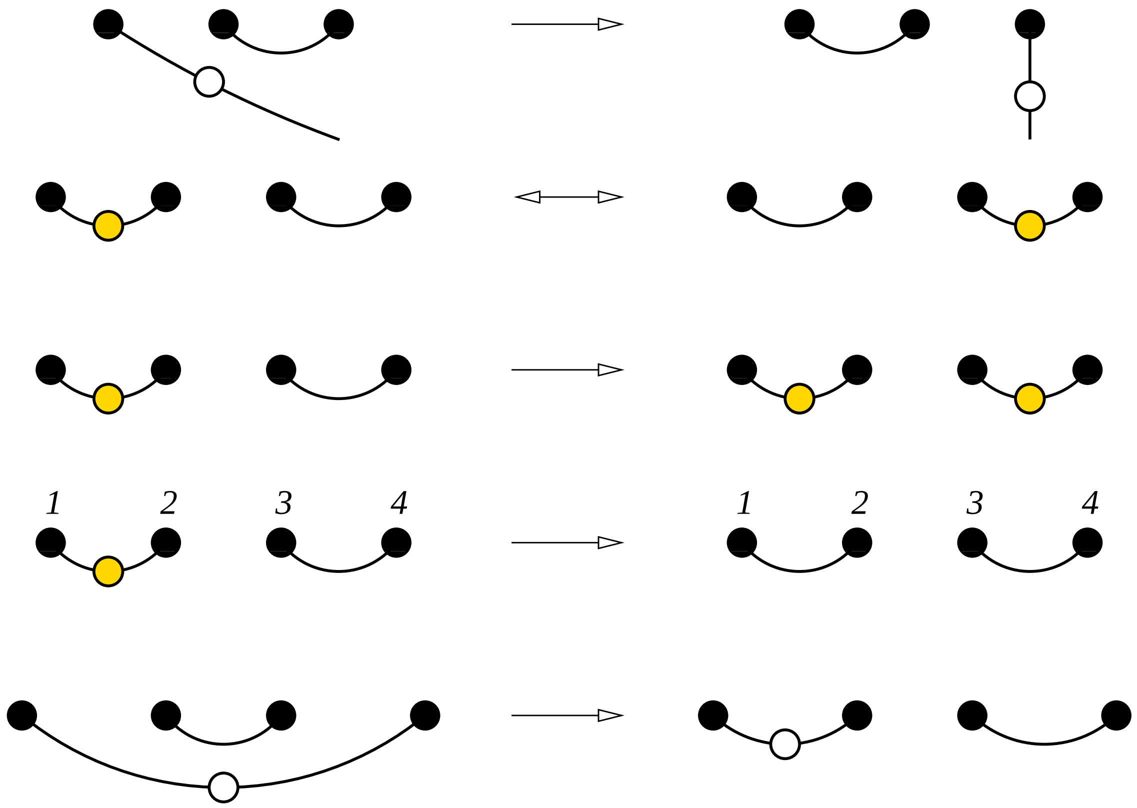
If nodes $p_1 + 1$ and $p_1 + 2$ in the north face are connected by a (necessarily undecorated) edge E', then $U_{p_1} D$ is the H-admissible diagram obtained by removing E', disconnecting E from the north face and reconnecting it to node $p_1 + 2$ in the north face, and installing a new undecorated edge between points p_1 and $p_1 + 1$ in the north face. The edge corresponding to E retains its original decoration status.

Lemma 3.2.3. *Assume that in the north face of D, nodes i and $i+1$ are connected by a decorated edge, e_1, and nodes $i + 2$ and $i + 3$ are connected by an undecorated edge, e_2. Assume also that $i > 1$. Then $U_i U_{i+1} D$ is the H-admissible diagram obtained from D by exchanging e_1 and e_2. This procedure has an inverse, since $D = U_{i+2} U_{i+1} U_i U_{i+1} D$.*

Lemma 3.2.4. *Assume that in the north face of D, nodes 1 and 2 are connected by a decorated edge, and nodes 3 and 4 are connected by an undecorated edge. Then the H-admissible diagram $\rho(\alpha) D$ is that obtained from D by decorating the edge connecting nodes 3 and 4.*

Lemma 3.2.5. *Assume that in the north face of D, nodes 1 and 2 are connected by a decorated edge, E, and nodes 3 and 4 are connected by an undecorated edge. Then the H-admissible diagram $U_3 \rho(\zeta) D$ is that obtained from D by removing the decoration on E.*

FIGURE 8. Respective illustrations of lemmas 3.2.2 to 3.2.6



Lemma 3.2.6. *Assume that in the north face of D, nodes i and $i+1$ are connected by an undecorated edge, e_1, and nodes $j < i$ and $k > i+1$ are connected by an edge, e_2. Assume also that j and k are chosen such that $|k-j|$ is minimal. Then D is of the form $U_i D'$, where D' is an H-admissible diagram which is the same as D except as regards the edges connected to nodes $j, i, i+1, k$ in the north face. Nodes j and i in D' are connected to each other by an edge with the same decoration as e_2, and nodes $i+1$ and k are connected to each other by an undecorated edge.*

As an illustration of what is going on, we present diagrammatic versions of these lemmas in Figure 8. A hollow circle indicates the site of an optional decoration.

Remark 3.2.7. The algebra Δ_{n+1} has an anti-automorphism, $*$, which reflects each diagram in the east-west line. Therefore, all of the five previous lemmas have corresponding statements about the south faces.

Using these five lemmas, we can prove the following result.

Proposition 3.2.8. *Any H-admissible diagram D with $n+1$ strands $(n \geq 2)$ can be written as a word in the images under ρ of the generating set G_n.*

Proof. The case $n = 2$ is done by Lemma 3.2.1.

Iteration of Lemma 3.2.6 reduces the consideration to diagrams D where all the non-propagating edges connect adjacent points.

We restrict ourselves to the nontrivial case where D has $r > 0$ non-propagating edges.

First, we assume that D has a propagating edge.

We define the diagram D_0, depending on D, which is chosen from the eight diagrams of form

$$D_0 = GU_4U_6 \cdots U_{2r-2}U_{2r},$$

where G is one of the nonidentity diagrams for the case $n = 2$ (see Lemma 3.2.1), and D_0 and D share the following three properties.

1. If D has a propagating edge meeting node 1 in the north face, then so does D_0.
2. If D has a propagating edge meeting node 1 in the south face, then so does D_0.
3. The leftmost propagating edges in D and D_0 are both decorated, or both undecorated.

If D does not have a propagating edge, we define

$$D_0 = U_1U_3 \cdots U_n,$$

where n is necessarily odd, and D_0 has no propagating edges.

We claim that $D = w_1D_0w_2$, where w_1 is a word in the generators obtained by the moves arising from lemmas 3.2.2 to 3.2.5 as stated, and w_2 is similar but arises from the reflected versions of these lemmas after applying $*$ (see Remark 3.2.7).

For reasons of symmetry, we concentrate on the word w_1, the other part being similar. To do this, we show that there is a diagram D' whose top half is that of D and whose bottom half is that of D_0, satisfying $D' = w_1D_0$. If D' has a propagating edge, then the leftmost one has the same decorated status as that of D or D_0.

We start with the diagram D_0. The first stage is to move the propagating edge (if there is one) so that it meets the north face at the desired point. This is achieved by iterations of Lemma 3.2.2.

Next, we generate all the decorated, non-propagating edges we desire, using Lemma 3.2.4. (Note that if we have to do this, then D_0 has a decorated edge connecting nodes 1 and 2 in the north face.) After these edges are formed, we commute them out of the way to the east using Lemma 3.2.3. If we require nodes 1 and 2 in D to be connected by an undecorated edge, this can be arranged by using Lemma 3.2.5 once. (Note that the definition of H-admissible implies that D must have two other points connected by a non-decorated edge in this case, so this is possible.) We then reach the diagram D' by further applications of Lemma 3.2.3.

The proof now follows. $\square$

3.3 Δ_{n+1} as a cellular algebra.

In order to count the dimension of Δ_{n+1}, it helps first to understand its structure as a cellular algebra. One can then compare the sizes of the cells to those arising from $TL(H_n)$ as in [**1**, Proposition 7.3.2].

It is convenient to introduce a dyadic notation for the H-admissible diagrams similar to that used for $TL(A_n)$ in [**13**, §5].

Definition 3.3.1. Let D be an H-admissible diagram for Δ_{n+1}. Remove all the propagating edges from D, then take the upper half of what remains and call it d_1. Invert the lower half of D in a horizontal line and call this d_2. Then D may be reconstituted from the ordered pair (d_1, d_2) provided that we know whether D has a decorated propagating edge or not.

We write $D = |d_1\rangle\langle d_2|$ if D has no decorated propagating edge, and $D = |d_1\rangle\langle d_2|^\bullet$ if D has a decorated propagating edge.

Lemma 3.3.2. *Let R be an integral domain of characteristic different from 2, 3 or 5 in which the polynomial x^2-x-1 splits into distinct linear factors $(x-\gamma_1)(x-\gamma_2)$.*

Writing γ for the image of x in the algebra $\Gamma = R[x]/\langle x^2 - x - 1\rangle$, we have

$$(\gamma - \gamma_1)^2 = (1 - 2\gamma_1)(\gamma - \gamma_1) \neq 0,$$

and a similar identity holds for $\gamma - \gamma_2$.

Proof. This is immediate. Note that $1-2\gamma_1 \neq 0$ because we are not in characteristic 5. $\square$

Definition 3.3.3. Let R satisfy the hypotheses of Lemma 3.3.2, and let $|d_1\rangle\langle d_2|$ be an H-admissible diagram. Then we define

$$|d_1\rangle\langle d_2|_1 := |d_1\rangle\langle d_2|^\bullet - \gamma_1 |d_1\rangle\langle d_2|$$

and

$$|d_1\rangle\langle d_2|_2 := |d_1\rangle\langle d_2|^\bullet - \gamma_2 |d_1\rangle\langle d_2|.$$

We recall the definition of a cellular algebra from [**5**]:

Definition 3.3.4. Let R be a commutative ring with identity. A *cellular algebra* over R is an associative unital algebra, A, together with a cell datum $(\Lambda, M, C, *)$ where

1. Λ is a poset. For each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set (the set of "tableaux" of type λ) such that
$$C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \to A$$
is injective with image an R-basis of A.
2. If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we write $C(S,T) = C^\lambda_{S,T} \in A$. Then $*$ is an R-linear involutory anti-automorphism of A such that $(C^\lambda_{S,T})^* = C^\lambda_{T,S}$.
3. If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for all $a \in A$ we have
$$a.C^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \mod A(< \lambda),$$
where $r_a(S', S) \in R$ is independent of T and $A(< \lambda)$ is the R-submodule of A generated by the set
$$\{C^\mu_{S'',T''} : \mu < \lambda, S'' \in M(\mu), T'' \in M(\mu)\}.$$

We now define our versions of the sets in the above definition.

Let Λ be the set of symbols $\{0\} \cup \{1, 2, \ldots, k, 1^\bullet, 2^\bullet, \ldots, k^\bullet\}$, where k is a natural number such that $k < (n+1)/2$, together with the symbol $(n+1)/2$ if n is odd. We put a partial order $<$ on these symbols by declaring that $i < j$ if $|i| > |j|$, where $|i| = i$ if i is a natural number, and $|i^\bullet| = i$.

If $\lambda \in \Lambda$, the set $M(\lambda)$ has elements parametrised by the half-diagrams $|d_1\rangle$ arising from H-admissible diagrams with $|\lambda|$ non-propagating edges in each half of the diagram.

The antiautomorphism $*$ corresponds to top-bottom inversion of an H-admissible diagram.

The map C takes elements d_1 and d_2 from $M(\lambda)$ and produces the element $C(d_1, d_2)$ which is defined to be

$$|d_1\rangle\langle d_2|_1$$

if λ is a natural number or

$$|d_1\rangle\langle d_2|_2$$

otherwise, unless $\lambda = 0$ or $\lambda = (n+1)/2$, in which case $C(d_1, d_2)$ is given by

$$|d_1\rangle\langle d_2|.$$

Note that the identity element appears in the image of C.

Theorem 3.3.5. *Let R be a ring satisfying the hypotheses of Lemma 3.3.2. Then the algebra Δ_{n+1} over the ring $R[v, v^{-1}]$ has a cell datum $(\Lambda, M, C, *)$, where the sets are given as above.*

Proof. The proof is largely straightforward. The fact that γ_1 and γ_2 are distinct ensures that the image of C is a basis for Δ_{n+1}.

The only other nontrivial part is the verification of axiom 3. Consider the product of two basis elements B_1 and B_2 parametrised by the respective elements λ and λ' of Λ. The only difficulty arises when $0 < \lambda, \lambda' < \frac{n+1}{2}$, so we concentrate on this case. It is convenient to think of each of the diagrams B_1 and B_2 as having a propagating edge decorated by one of the elements $\gamma - \gamma_1$ or $\gamma - \gamma_2$ of Γ, where an ordinary decorated edge is thought of as being decorated by $\gamma \in \Gamma$, and an undecorated one as being decorated by $1 \in \Gamma$. (Note that the third relation in Figure 5 corresponds to the equation $\gamma^2 = \gamma + 1$.)

We will assume that $|\lambda'| \geq |\lambda|$; the other case is similar. Let us define γ_i by saying that the propagating edge of B_2 carries the element $\gamma - \gamma_i$ (where $i \in \{1, 2\}$). If the product $B = B_1 B_2$ is a tangle with strictly fewer propagating edges than B_2 (and therefore fewer than B_1, since $|\lambda'| \geq |\lambda|$) then it is clear that B is a linear combination of basis elements corresponding to elements $r \in \Lambda$ with $r < \lambda'$, so axiom 3 holds.

The other possibility is that the product B has the same number of propagating edges as B_2 and, furthermore, that the leftmost propagating edge, E, of B contains (as a segment) the leftmost propagating edge of B_2. The edge E therefore carries the generalized decoration $\gamma - \gamma_i$, and possibly other decorations of various kinds.

Lemma 3.3.2 shows that if we multiply together all the decorations on the edge E (where an ordinary decoration corresponds to γ, as before), we obtain a (possibly zero) multiple of $\gamma - \gamma_i$. If we obtain zero then the product B is zero and there is nothing more to prove. Otherwise, B is a linear combination of basis elements whose leftmost propagating edges all carry $\gamma - \gamma_i$, namely a combination corresponding to the element $\lambda' \in \Lambda$. Since the structure constants are not affected by the pattern of non-propagating edges in the south face of B_2, axiom 3 follows. $\square$

3.4 Faithfulness of the diagram representation.

Using the results of §3.3, we can enumerate the number of H-admissible diagrams of various types.

Lemma 3.4.1. *The size of the set $M(\lambda)$ associated with the algebra Δ_{n+1} is equal to*

$$\binom{n+1}{|\lambda|} - 1,$$

unless $|\lambda| = 0$ in which case the set has size 1.

Proof. If we generalized the H-admissible diagrams by excluding parts (i) and (ii) of Definition 2.2.1, then it would follow from [**10**, Proposition 2] that there would be $\binom{n+1}{k}$ half diagrams with k non-propagating edges. If $k > 0$ then the force of part (ii) of Definition 2.2.1 is to exclude just one element: the one with an undecorated edge connecting points 1 and 2 and a decorated edge connecting points $2m+1$ and $2m+2$ for $m < k$ (see Figure 4). This proves the assertion for $|\lambda| > 0$, and the assertion for $|\lambda| = 0$ is trivial. $\square$

Theorem 3.4.2. *Working over $\mathcal{A}$, the ranks of Δ_{n+1} and $TL(H_n)$ are identical. Therefore, ρ is an isomorphism.*

Proof. Stembridge [**11**, §3.4] proves that the number of "fully commutative" elements in a Coxeter group of type H_n is given by

$$1 + \sum_{\lambda \in \Lambda, |\lambda| > 0} \left(\binom{n+1}{|\lambda|} - 1 \right)^2 = \binom{2n+2}{n+1} - 2^{n+2} + n + 3.$$

Graham [**4**, Theorem 6.2] shows that these fully commutative elements index a basis for $TL(H_n)$. It follows from Lemma 3.4.1 that the rank of Δ_{n+1} is the same as the rank of $TL(H_n)$. The fact that ρ is an isomorphism follows from Proposition 3.1.2 and Proposition 3.2.8. □

Remark 3.4.3. The fact that $TL(H_n)$ is cellular if x^2-x-1 splits has been observed by Graham [**4**, Remark 9.8], although a cell datum is not explicitly given.

Remark 3.4.4. The similarity with the blob algebra of [**10**] which is touched upon in the proof of Lemma 3.4.1 goes further. If conditions (i) and (ii) of Definition 2.2.1 are dropped, then the resulting algebra is isomorphic to the algebra of [**10**], although the isomorphism is not canonical.

4. Applications

We now examine some applications of theorems 3.3.5 and 3.4.2.

4.1 Positivity Properties.

We have seen how the diagram basis for $TL(H_n)$ can be expressed as monomials in a certain set of algebra generators (this follows from Proposition 3.2.8 and Theorem 3.4.2). We now show that this diagram basis has a positivity property.

Proposition 4.1.1. *Assume we are working over the ring $\mathcal{A}$.*

Any basis element occurring with nonzero coefficient in the product of two basis elements D_1D_2 associated with the respective elements λ_1 and λ_2 of Λ occurs with coefficient $c[2]^k$, where c is a positive integer and $k \leq \max(|\lambda_1|, |\lambda_2|)$.

In particular, the structure constants of the basis of diagrams are polynomials in $\mathbb{N}[v, v^{-1}]$.

Proof. Note that the simplification process given by the rules in Figure 5 preserves positivity, and that $[2]$ is a polynomial in $\mathbb{N}[v, v^{-1}]$.

It is immediate that there cannot be more loops forming in the diagram multiplication than there were non-propagating edges in each half of either D_1 or D_2, which proves the assertion about the number k. □

Note that any basis obtained from monomials in the original set of generators cannot have this property: consider the monomial $E_1E_2E_1E_2E_1$.

4.2 Semisimplicity.

We recall from the theory of cellular algebras in [**5**, §2] that there is a bilinear form $\phi_\lambda(d_1, d_2) = \langle d_1, d_2 \rangle$ on the cell module $W(\lambda)$. (Recall that the module $W(\lambda)$ has a basis parametrised by the elements of $M(\lambda)$ for a fixed λ.) This form is defined from the equation

$$C(e_1, d_1)C(d_2, e_2) = \langle d_1, d_2 \rangle C(e_1, e_2) \mod A(< \lambda).$$

This is independent of the choice of e_1 and e_2, where e_1, e_2, d_1, d_2 are all elements of $M(\lambda)$ for the same λ.

The following result proves [**1**, Conjecture 7.3.1].

Theorem 4.2.1. *Let R be a field satisfying the hypotheses of Lemma 3.3.2. Then the algebra $TL(H_n)$ over R is semisimple, and all the cell modules $W(\lambda)$ are irreducible and pairwise inequivalent.*

Proof. It is enough by [**5**, Theorem 3.8] to prove that ϕ_λ is nondegenerate for each λ.

Choose an element $\lambda \in \Lambda$ and two elements $d_1, d_2 \in M(\lambda)$, where possibly $d_1 = d_2$. Now consider $v^{-|\lambda|}\phi_\lambda(d_1, d_2) = v^{-|\lambda|}\langle d_1, d_2 \rangle$.

It follows from Proposition 4.1.1 that $v^{-|\lambda|}\langle d_1, d_2 \rangle$ is a polynomial in v^{-1}; furthermore, the constant term of the polynomial is zero unless $d_1 = d_2$. To see this, we use the fact that loops carrying a single blob result in annihilation of the associated diagram, and loops which carry 0 or 2 blobs both correspond to multiplication by [2]. In the case where $d_1 = d_2$, Lemma 3.3.2 shows that the constant term of the polynomial is nonzero and equal to $(1 - 2\gamma_1)$ or $(1 - 2\gamma_2)$, depending on the λ which we are considering.

We have now constructed an almost orthogonal basis (i.e. orthogonal modulo the span of strictly negative powers of v) for the module $W(\lambda)$ with respect to ϕ_λ. It follows that ϕ_λ is nondegenerate, as required. $\square$

Note that Lemma 3.4.1 now tells us the dimensions of the irreducible modules. This confirms [**1**, Conjecture 7.3.3].

4.3 Branching rules.

In this section, we continue to assume that the base ring of $TL(H_n)$ is a field in which $x^2 - x - 1$ splits into distinct linear factors, although we do not assume that $TL(H_n)$ is semisimple. The diagram calculus we have developed allows us to study the behaviour of the cell modules $W(\lambda)$ for $TL(H_n)$ upon restriction to $TL(H_{n-1})$ (assuming n is at least 3). The embedding of $TL(H_{n-1})$ into $TL(H_n)$ is the natural one arising from the identification of the algebra generators, or the addition of a vertical edge on the east of the diagram. To describe the branching rules, it is convenient to make the following definition.

Definition 4.3.1. Let $\lambda \in \Lambda = \Lambda(H_n)$, and suppose that $0 < |\lambda| < \frac{n+1}{2}$. We define an element $\lambda - 1 \in \Lambda(H_{n-1})$ as follows:

$$\lambda - 1 := \begin{cases} 0 & \text{if } |\lambda| = 1, \\ (i-1)^\bullet & \text{if } |\lambda| \neq 1,\ \lambda = i^\bullet \text{ where } i \in \mathbb{N}, \text{ and } i - 1 \neq n/2, \\ i - 1 & \text{otherwise.} \end{cases}$$

Proposition 4.3.2. *Let $W(\lambda, n)$ be a cell module for $TL(H_n)$. Then, after restriction to $TL(H_{n-1})$, $W(\lambda, n)_{n-1}$ has a filtration by the cell modules $W(\lambda', n-1)$ of $TL(H_{n-1})$ described as follows.*

If $\lambda = 0$ then restriction gives the trivial module corresponding to the poset element 0.

If $|\lambda| = 1$ then the composition factors occurring correpond to the poset elements 0 and λ, each with multiplicity 1.

If $\lambda = \frac{n+1}{2}$ then the composition factors occurring correspond to the poset elements 0, $\frac{n-1}{2}$ and $\left(\frac{n-1}{2}\right)^\bullet$, each with multiplicity 1.

For other values of λ, the composition factors occurring correspond to the poset elements 0, $\lambda - 1$ and λ, each with multiplicity 1.

Proof. We first tackle the fourth case, dealing with the general value of λ.

The key observation, which is familiar from the diagram calculi of other types, is as follows. The half-diagrams in $M(\lambda)$ which have a non-connected point at the eastern extreme form a submodule for $TL(H_{n-1})$ on restriction, corresponding to removal of the easternmost point. This is canonically isomorphic to the module corresponding to λ in $\Lambda(H_{n-1})$.

The quotient module associated with this submodule is obtained by taking the other elements of $M(\lambda)$ and, for each one, removing the easternmost point and the edge connected to it. However, this is not the same as one of the cell modules for $TL(H_{n-1})$, because an inadmissible half-diagram occurs. (All the edges in this are decorated, except for the one connecting points 1 and 2.) The reason that this arises is that in the original diagram, the two easternmost points could have been connected by an undecorated edge, making the half-diagram admissible, but this edge was removed in the restriction process. The admissible diagrams arising from this procedure span a cell module isomorphic to that parametrised by $\lambda - 1$. The appearance of the inadmissible diagram corresponds to a top quotient isomorphic to the trivial module.

The cases $|\lambda| = 0$ and $|\lambda| = 1$ can be obtained by degenerate versions of this technique.

The case $\lambda = \frac{n+1}{2}$ is the most subtle. To deal with it, it is convenient to modify the basis for the cell module $W(\lambda, n)$. One of the half-diagrams in $M(\lambda)$ becomes inadmissible once the rightmost point and its associated edge have been removed; we denote this half-diagram by d_0.

The other half-diagrams fall naturally into pairs as we now describe. First, note that any edge of a half-diagram of $M(\lambda)$ (for $\lambda = \frac{n+1}{2}$) is exposed to the west face, since there are no propagating edges involved. There is therefore an involution on $M(\lambda)\backslash\{d_0\}$ given by changing the decorated status of the edge connected to the rightmost point. The orbits are all of size 2.

If d and $d^\bullet$ are two elements in the same orbit (where $d^\bullet$ carries the extra decoration), we define new basis elements d_1 and d_2 for $W(\lambda, n)$ by

$$d_1 := d^\bullet - \gamma_1 d$$

and

$$d_2 := d^\bullet - \gamma_2 d.$$

Analysis of the diagrams now shows that the diagram d_0 corresponds to a top quotient of $W(\lambda, n)$ isomorphic to the trivial module. Furthermore, the submodule of $W(\lambda, n)$ spanned by the new basis elements d_i breaks up as a direct sum: the span of the elements d_1 is canonically isomorphic to $W\left(\frac{n-1}{2}, n-1\right)$, and the span of the elements d_2 is canonically isomorphic to $W\left(\left(\frac{n-1}{2}\right)^\bullet, n-1\right)$. $\square$

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