FREELY BRAIDED ELEMENTS IN COXETER GROUPS, II

R.M. Green and J. Losonczy

Department of Mathematics
University of Colorado
Campus Box 395
Boulder, CO 80309-0395
USA
E-mail: rmg@euclid.colorado.edu

Department of Mathematics
Long Island University
Brookville, NY 11548
USA
E-mail: losonczy@e-math.ams.org

Abstract. We continue the study of freely braided elements of simply laced Coxeter groups, which we introduced in a previous work. A known upper bound for the number of commutation classes of reduced expressions for an element of a simply laced Coxeter group is shown to be achieved only when the element is freely braided; this establishes the converse direction of a previous result. It is also shown that a simply laced Coxeter group has finitely many freely braided elements if and only if it has finitely many fully commutative elements.

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Introduction

In [**5**] we defined, for an arbitrary simply laced Coxeter group, a subset of "freely braided elements". Such elements include the fully commutative elements of Stembridge [**8**] as a particular case. The idea behind the definition is that although it

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may be necessary to use long braid relations in order to pass between two reduced expressions for a freely braided element, the necessary long braid relations in a certain sense do not interfere with one another.

Every reduced expression for a Coxeter group element w determines a total ordering of the set of positive roots made negative by w. These totally ordered sets are known as “root sequences”. If a root sequence for an element w of a simply laced Coxeter group contains a consecutive subsequence of the form $\alpha, \alpha+\beta, \beta$, then we refer to the set of these roots as a “contractible inversion triple” of w. A group element is said to be “freely braided” if its contractible inversion triples are pairwise disjoint.

Let $N(w)$ denote the number of contractible inversion triples of w. It was shown in [**5**] that the number of commutation classes (short braid equivalence classes of reduced expressions) of w is bounded above by $2^{N(w)}$, and that this bound is achieved if w is freely braided. In this paper, we prove that the bound can be achieved only for freely braided w (Theorem 2.2.1). This was previously known in the type A setting [**5**, Theorem 5.2.1], but the argument given here has the advantage of being conceptual as well as more general.

The second main result of this paper is a classification of the simply laced Coxeter groups having only finitely many freely braided elements. Note that it is possible for such a group to be infinite. It turns out (see the discussion following Theorem 3.3.3) that this question has the same answer as a classification question previously answered by others [**2**, **4**, **8**]: we will show in Theorem 3.3.3 that a simply laced Coxeter group has finitely many freely braided elements if and only if it has finitely many fully commutative elements. One implication here is easy, but the converse requires some effort. Our proof of Theorem 3.3.3 does not rely on a case analysis based on any classification result.

1. Preliminaries

1.1 Basic terminology and notation.

Let W be a simply laced Coxeter group with distinguished generators $S = \{s_i : i \in I\}$ and Coxeter matrix $(m_{ij})_{i,j \in I}$. For the basic facts concerning Coxeter groups, see [**1**] or [**6**]. Denote by I^* the free monoid on I. We call the elements of I *letters* and those of I^* *words*. The *length* of a word is the number of factors required to write the word as a product of letters. Let $\phi : I^* \longrightarrow W$ be the surjective morphism of monoid structures satisfying $\phi(i) = s_i$ for all $i \in I$. A word $\mathbf{i} \in I^*$ is said to *represent* its image $w = \phi(\mathbf{i}) \in W$; furthermore, if the length of $\mathbf{i}$ is minimal among the lengths of all the words that represent w, then we call $\mathbf{i}$ a *reduced expression* for w. The *length* of w, denoted by $\ell(w)$, is then equal to the length of $\mathbf{i}$.

Let V be a vector space over the field of real numbers with basis $\{\alpha_i : i \in I\}$, and denote by B the *Coxeter form* on V associated to W. This is the symmetric bilinear form satisfying $B(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m_{ij}}$ for all $i, j \in I$. We view V as the underlying space of a reflection representation of W, determined by the equalities $s_i \alpha_j = \alpha_j - 2B(\alpha_j, \alpha_i)\alpha_i$ for all $i, j \in I$. The Coxeter form is preserved by W relative to this representation.

Denote by Φ the *root system* of W, i.e., the set $\{w\alpha_i : w \in W \text{ and } i \in I\}$. Let Φ^+ be the set of all $\beta \in \Phi$ such that β is expressible as a linear combination of the α_i with nonnegative coefficients, and let $\Phi^- = -\Phi^+$. We have $\Phi = \Phi^+ \cup \Phi^-$ (disjoint). The elements of Φ^+ (respectively, Φ^-) are called *positive* (respectively, *negative*) roots. The α_i are also referred to as *simple* roots. We define the *height* of any root β to be the sum of the coefficients used to express β as a linear combination of the simple roots.

Associated to each $w \in W$ is the *inversion set* $\Phi(w) = \Phi^+ \cap w^{-1}(\Phi^-)$. It has $\ell(w)$ elements and determines w uniquely. Given any reduced expression $i_1 i_2 \cdots i_n$ for w, we have $\Phi(w) = \{r_1, r_2, \ldots, r_n\}$, where $r_1 = \alpha_{i_n}$ and $r_l = s_{i_n} \cdots s_{i_{n-l+2}}(\alpha_{i_{n-l+1}})$ for all $l \in \{2, \ldots, n\}$. Form the sequence $\bar{r} = (r_1, r_2, \ldots, r_n)$. We call $\bar{r}$ the *root* *sequence* of $i_1 i_2 \cdots i_n$, or a root sequence *for* w. Notice that any initial segment of a root sequence is also a root sequence for some element of W.

Let $w \in W$. Any subset of $\Phi(w)$ of the form $\{\alpha, \beta, \alpha+\beta\}$ will be called an *inversion triple* of w. We say that an inversion triple T of w is *contractible* if there is a root sequence for w in which the elements of T appear consecutively (in some order). The number of contractible inversion triples of w will be denoted by $N(w)$. If the contractible inversion triples of w are pairwise disjoint, then w is said to be *freely braided.*

1.2 Braid moves.

Given any $i, j \in I$ and any nonnegative integer n, we write $(i, j)_n$ for the length n word $iji \cdots \in I^*$. Let $\mathbf{i}, \mathbf{j} \in I^*$ and let $i, j \in I$ with $m_{ij} \neq 1$. We call the substitution $\mathbf{i}(i, j)_{m_{ij}}\mathbf{j} \to \mathbf{i}(j, i)_{m_{ji}}\mathbf{j}$ a *braid move*, qualifying it *short* or *long* according as m_{ij} equals 2 or 3.

Let $w \in W$. A well-known result of Matsumoto [**7**] and Tits [**9**] states that any reduced expression for w can be transformed into any other by applying a (possibly empty) sequence of braid moves.

We say that two words are *commutation equivalent* if one can be transformed into the other by a sequence of short braid moves. The set of words that are commutation equivalent to a given word is called the *commutation class* of that word. If the set of reduced expressions for an element $w \in W$ forms a single commutation class, then we call w *fully commutative*, following [**8**, §1].

Applying a braid relation to a reduced expression corresponds to applying a permutation to the root sequence of that reduced expression. The following proposition makes this more precise.

Proposition 1.2.1 [**5**, Proposition 3.1.1]**.** *Let $w \in W$, let $\mathbf{i}, \mathbf{j} \in I^*$ and let $i, j, k \in I$. Denote the length of $\mathbf{j}$ by n.*

(a) *Assume that $\mathbf{i}ij\mathbf{j}$ is a reduced expression for w, and let $\bar{r} = (r_l)$ be the associated root sequence.*

(i) *Suppose* $m_{ij} = 2$, *so that* $\mathbf{i}ji\mathbf{j}$ *is also a reduced expression for* w. *Then the root sequence* $\bar{r}'$ *of* $\mathbf{i}ji\mathbf{j}$ *can be obtained from* $\bar{r}$ *by interchanging* r_{n+1} *and* r_{n+2}, *which are mutually orthogonal relative to* B.

(ii) *If* r_{n+1} *and* r_{n+2} *are orthogonal, then* $m_{ij} = 2$.

(b) *Assume that* $\mathbf{i}ijk\mathbf{j}$ *is a reduced expression for* w, *and let* $\bar{r} = (r_l)$ *be the associated root sequence.*

(i) *Suppose* $k = i$, *so that* $m_{ij} = 3$ *and* $\mathbf{i}jij\mathbf{j}$ *is also a reduced expression for* w. *Then the root sequence* $\bar{r}'$ *of* $\mathbf{i}jij\mathbf{j}$ *can be obtained from* $\bar{r}$ *by interchanging* r_{n+1} *and* r_{n+3}. *Furthermore, we have* $r_{n+1} + r_{n+3} = r_{n+2}$.

(ii) *If* $r_{n+1} + r_{n+3} = r_{n+2}$, *then* $k = i \neq j$ *and* $m_{ij} = 3$. $\square$

Let $\bar{r}$ and $\bar{r}'$ be as in part (a)(i) (respectively, part (b)(i)) of Proposition 1.2.1. Employing again the terminology used above for words, we say that the passage from $\bar{r}$ to $\bar{r}'$ is obtained by a *short braid move* (respectively, *long braid move*). Two root sequences are said to be *commutation equivalent* if one can be transformed into the other by applying a sequence of short braid moves. The set of root sequences that are commutation equivalent to a given root sequence is called the *commutation class* of that root sequence.

Let $w \in W$. The recipe for associating a root sequence to a reduced expression defines a bijection from the set of reduced expressions for w to the set of root sequences for w, and by Proposition 1.2.1, this bijection is compatible with the application of both long and short braid moves. Hence, by the result of Matsumoto and Tits cited above, any root sequence for w can be transformed into any other by applying a sequence of long and short braid moves. It also follows that there is a natural bijection between the set of commutation classes of reduced expressions for w and the set of commutation classes of root sequences for w.

A *subword* of a word $i_1 i_2 \cdots i_n$ (each $i_l \in I$) is any word of the form $i_p i_{p+1} \cdots i_q$, where $1 \leq p \leq q \leq n$.

Proposition 1.2.2. *Let* $w \in W$. *The following are equivalent:*

(i) *w is fully commutative.*

(ii) *w has no inversion triples.*

(iii) *w has no contractible inversion triples.*

(iv) *No reduced expression for w contains a subword of the form iji, where $i, j \in I$.*

Proof.

(i) $\Rightarrow$ (ii) This is the implication (a) $\Rightarrow$ (c) of [**3**, Theorem 2.4].

(ii) $\Rightarrow$ (iii) This is immediate from the definitions.

(iii) $\Rightarrow$ (iv) If w has a reduced expression with a subword of the form iji, then Proposition 1.2.1 (b)(i) shows that w has a contractible inversion triple.

(iv) $\Rightarrow$ (i) If w is not fully commutative, then there exists a pair of commutation inequivalent reduced expressions for w. It follows by the result of Matsumoto and Tits mentioned above that w has a reduced expression to which a braid move can be applied. Thus, w does not satisfy (iv). $\square$

2. Freely braided elements and commutation classes

2.1 The map F_w.

Let $w \in W$. Fix an arbitrary antisymmetric relation $\preceq$ on $\Phi(w)$ with the property that any two roots in $\Phi(w)$ are comparable relative to $\preceq$. Let $\mathcal{C}(w)$ and $\mathcal{I}(w)$ denote the set of commutation classes of root sequences for w and the set of contractible inversion triples of w, respectively. We define a map

$$F_w : \mathcal{C}(w) \longrightarrow \{0, 1\}^{\mathcal{I}(w)},$$

depending on $\preceq$, as follows. If $\mathcal{I}(w)$ is empty, then $\{0, 1\}^{\mathcal{I}(w)}$ contains just the empty map, and the set $\mathcal{C}(w)$ is also a singleton by Proposition 1.2.2. Thus, in this situation, F_w is uniquely determined. Suppose that $\mathcal{I}(w)$ is nonempty. Let $C \in \mathcal{C}(w)$ and let $\leq_C$ be the partial ordering of $\Phi(w)$ obtained by taking the transitive closure of the following relations: $\alpha < \beta$ whenever α lies to the left of β in some root sequence from C and $B(\alpha, \beta) \neq 0$. Note that $\leq_C$ is well-defined by [**5**, Proposition 3.1.5]. Given any $\{\alpha, \beta, \alpha+\beta\} \in \mathcal{I}(w)$, we define $F_w(C)(\{\alpha, \beta, \alpha+\beta\})$ to be 0 if α and β are in the same relative order with respect to $\leq_C$ and $\preceq$, and otherwise we define $F_w(C)(\{\alpha, \beta, \alpha+\beta\})$ to be 1.

The map F_w is injective [**5**, Theorem 4.1.1].

It will be convenient to have the following terminology when determining the surjectivity, or otherwise, of F_w. Let $w \in W$ and let $\mathcal{T}$ be a subset of $\mathcal{I}(w)$. We say that F_w *separates* $\mathcal{T}$ if every map from $\mathcal{T}$ to $\{0, 1\}$ is the restriction of some element of $F_w(\mathcal{C}(w))$. Clearly, if F_w fails to separate some nonempty subset of $\mathcal{I}(w)$, then F_w is not surjective.

2.2 First main result.

It was shown in [**5**, Corollary 4.1.2, Corollary 4.2.4] that every $w \in W$ has at most $2^{N(w)}$ commutation classes, with equality if w is freely braided. The following theorem shows that equality is achieved only if w is freely braided. For the special case where W is of type A, this was already done in [**5**, Theorem 5.2.1] using an ad hoc argument.

Theorem 2.2.1. *If $w \in W$ has $2^{N(w)}$ commutation classes, then w is freely braided.*

Proof. Let w be a non-freely-braided element. Since F_w is injective, it suffices to prove that F_w is not surjective. Let α be a root belonging to at least two contractible inversion triples of w, and assume that the height of α is maximal with respect to this property. Let T and T' be distinct contractible inversion triples of w containing α. Note that $|T \cap T'| = 1$ (this follows easily from [**5**, Remark 2.2.2] and the contractibility of the triples). By symmetry, there are three cases to consider.

Case 1: $T = \{\alpha, \beta, \alpha+\beta\}$ and $T' = \{\alpha, \gamma, \alpha+\gamma\}$.

By [**5**, Remark 2.2.2] and the contractibility of T, w has a root sequence of the form

$$(\ldots, \alpha, \alpha+\beta, \beta, \ldots, \alpha+\gamma, \ldots, \gamma, \ldots)$$

or

$$(\ldots, \gamma, \ldots, \alpha+\gamma, \ldots, \alpha, \alpha+\beta, \beta, \ldots).$$

We assume the existence of a sequence of the former type, the argument for the latter being similar. By our choice of α, the roots $\alpha+\beta$ and $\alpha+\gamma$ cannot belong to the same contractible inversion triple of w. Suppose that $\alpha+\beta$ is not orthogonal to $\alpha+\gamma$. Then, by [**5**, Proposition 3.2.2], $\alpha+\beta$ lies to the left of $\alpha+\gamma$ in every root sequence for w, and so it is impossible for α to be at the same time to the left of $\alpha+\beta$ and to the right of $\alpha+\gamma$. Thus, F_w does not separate $\{T, T'\}$.

Suppose instead that $\alpha+\beta$ is orthogonal to $\alpha+\gamma$. Then $\alpha+\beta$ and γ are not mutually orthogonal, since $\alpha+\beta$ is not orthogonal to α. Furthermore, $\alpha+\beta$ and γ cannot belong to the same contractible inversion triple of w, by our choice of α. It follows (again by [**5**, Proposition 3.2.2]) that $\alpha+\beta$ lies to the left of γ in every root sequence for w. This means that in any root sequence for w in which α lies to the left of $\alpha+\beta$, the root γ necessarily lies to the right of $\alpha+\gamma$ (otherwise, γ lies to the left of $\alpha+\gamma$, which must then be to the left of α, which in turn is to the left of $\alpha+\beta$, a contradiction). Again, F_w does not separate $\{T, T'\}$.

Case 2: $T = \{\alpha, \beta, \alpha-\beta\}$ and $T' = \{\alpha, \gamma, \alpha-\gamma\}$.

Here, we may assume without loss of generality that w has a root sequence of the form

$$(\ldots, \gamma, \ldots, \beta, \alpha, \alpha-\beta, \ldots, \alpha-\gamma, \ldots).$$

Note that γ cannot be orthogonal to both β and $\alpha-\beta$, or it would be orthogonal to their sum. We may assume that γ is not orthogonal to β. If no contractible inversion triple of w contains both γ and β, then γ lies to the left of β in every root sequence for w, and consequently there is no root sequence for w in which β lies to the left of α and γ lies to the right of α. It follows that F_w does not separate $\{T, T'\}$.

Suppose instead that there is a contractible inversion triple T'' of w that contains

γ and β. We claim that F_w does not separate $\{T, T', T''\}$. To see this, observe that if C is a commutation class relative to which γ lies to the left of β (this determines $F_w(C)(T'')$), and β lies to the left of α (this determines $F_w(C)(T)$), then γ lies to the left of α (so that $F_w(C)(T')$ is also determined). (These conditions are well-defined by [**5**, Proposition 3.1.5].)

Case 3: $T = \{\alpha, \beta, \alpha - \beta\}$ and $T' = \{\alpha, \gamma, \alpha + \gamma\}$.

In this situation, w has a root sequence of the form

$$(\ldots, \beta, \alpha, \alpha - \beta, \ldots, \alpha + \gamma, \ldots, \gamma, \ldots)$$

or

$$(\ldots, \gamma, \ldots, \alpha + \gamma, \ldots, \beta, \alpha, \alpha - \beta, \ldots).$$

We deal with the former sequence, the analysis of the latter being similar.

Note that $\alpha + \gamma$ cannot be orthogonal to both β and $\alpha - \beta$, or it would be orthogonal to their sum. We may assume that $\alpha + \gamma$ is not orthogonal to β. By our choice of α, the roots $\alpha + \gamma$ and β cannot belong to the same contractible inversion triple of w. Hence, β lies to the left of $\alpha + \gamma$ in every root sequence for w. It follows that there is no root sequence for w in which α lies to the right of $\alpha + \gamma$ and β lies to the right of α. This means that F_w does not separate $\{T, T'\}$.

We conclude that F_w is not surjective. $\square$

Corollary 2.2.2. *Every $w \in W$ has at most $2^{N(w)}$ commutation classes, with equality if and only if w is freely braided.* $\square$

3. Free braidedness and full commutativity

The goal of this section, achieved by Theorem 3.3.3, is to prove that W has finitely many freely braided elements if and only if it has finitely many fully commutative elements.

3.1 Reduced expressions for freely braided elements.

For the purposes of the proof of Theorem 3.3.3, we wish to have a clearer picture on the nature of reduced expressions for freely braided elements.

Definition 3.1.1. Let $\mathbf{i}$ be a word in I^* and suppose that $\mathbf{i}$ can be written as $\mathbf{u}_0\mathbf{b}_1\mathbf{u}_1\mathbf{b}_2\mathbf{u}_2\cdots\mathbf{b}_p\mathbf{u}_p$, where each $\mathbf{b}_l$ is of the form iji for some $i, j \in I$ with $m_{ij} = 3$. Then we call $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ a *braid sequence* for $\mathbf{i}$. If, furthermore, $\mathbf{i}$ is reduced and $w = \phi(\mathbf{i})$ is freely braided, then we say that $\mathbf{i}$ is *contracted* provided there exists a braid sequence for $\mathbf{i}$ with $p = N(w)$ terms.

Definition 3.1.2. Let $\mathbf{i} \in I^*$. A word $\mathbf{j} \in I^*$ is said to be *close* to $\mathbf{i}$ if there is a (possibly empty) braid sequence $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ for $\mathbf{i}$ such that $\mathbf{j}$ is the word obtained by applying a long braid move to each of the $\mathbf{b}_l$. We also say that $\mathbf{j}$ is close to $\mathbf{i}$ *via the sequence* $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$.

Note that if two words are close to one another, then they represent the same element of W. Note also that any expression that is close to a contracted reduced expression is itself contracted.

We say that $i, j \in I$ are *m-commuting*, or simply *commuting*, if $m_{ij} \neq 3$.

Proposition 3.1.3. *Let $w \in W$ be freely braided.*

(i) *There exists a contracted reduced expression* $\mathbf{i}$ *for w.*

(ii) *The reduced expressions close to* $\mathbf{i}$*, which are also contracted reduced expressions, form an irredundantly described set of commutation class representatives for w.*

(iii) *Any contracted reduced expression for w has a unique braid sequence with $N(w)$ terms.*

Proof. By [**5**, Theorem 4.2.3], there is a root sequence $\bar{r}$ for w such that the roots in any given contractible inversion triple of w appear consecutively in $\bar{r}$. Part (i) follows by applying Proposition 1.2.1 (b)(ii): take $\mathbf{i}$ to be the reduced expression corresponding to $\bar{r}$, and note that the $N(w)$ contractible inversion triples correspond to a braid sequence for $\mathbf{i}$ with $N(w)$ terms. In view of Proposition 1.2.1 (b)(i), the expression $\mathbf{i}$, or any contracted reduced expression for w, has at most one braid sequence with $N(w)$ terms. This proves (iii).

If $\mathbf{i}', \mathbf{i}'' \in I^*$ are distinct and close to $\mathbf{i}$, then $\mathbf{i}'$ is not commutation equivalent to $\mathbf{i}''$ (to see this, observe that the sequence of occurrences of any pair of non-m-commuting letters in a word is an invariant of the commutation class of that word). Therefore, since there are $2^{N(w)}$ expressions close to $\mathbf{i}$, and since w has exactly $2^{N(w)}$ commutation classes (by Corollary 2.2.2), part (ii) is proved. $\square$

Remark 3.1.4. Let $w \in W$. From the proof of Proposition 3.1.3 (ii), we see that if a reduced expression for w has a braid sequence with p terms, then w has at least 2^p commutation classes.

Definition 3.1.5. Let $\mathbf{i}$ be a contracted reduced expression for a freely braided element $w \in W$, and write

$$\mathbf{i} = \mathbf{u}_1\mathbf{b}_1\mathbf{u}_2\mathbf{b}_2 \cdots \mathbf{u}_{N(w)}\mathbf{b}_{N(w)}\mathbf{q},$$

where $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{N(w)}$ is the unique braid sequence for $\mathbf{i}$ with $N(w)$ terms. If $N(w) > 0$, then we define $D(\mathbf{i}) = D^1(\mathbf{i})$ to be the word in I^* obtained from $\mathbf{i}$ by deleting the rightmost letter in $\mathbf{b}_{N(w)}$. We do not define $D(\mathbf{i})$ if $N(w) = 0$. By induction, we write $D^n(\mathbf{i})$ for $D(D^{n-1}(\mathbf{i}))$ if $n > 1$ and the composition is defined. We agree that $D^0(\mathbf{i}) = \mathbf{i}$, regardless of the value of $N(w)$.

Our strategy for the proof of Theorem 3.3.3 will be to argue that if $\mathbf{i}$ is a contracted reduced expression for a freely braided element $w \in W$, then $D^{N(w)}(\mathbf{i})$ is a well-defined reduced expression for some fully commutative element. This will require several intermediate steps. One of these is the following technical lemma.

Lemma 3.1.6. *Maintain the notation of Definition 3.1.5. Suppose that $D(\mathbf{i})$ is a reduced expression for a freely braided element $w' \in W$ with $N(w') = N(w) - 1$. Then any expression close to $D(\mathbf{i})$ is of the form $D(\mathbf{j})$, where $\mathbf{j}$ is a reduced expression for w that is close to $\mathbf{i}$ via a braid sequence not involving $\mathbf{b}_{N(w)}$.*

Proof. The hypotheses on $D(\mathbf{i})$ imply that it is a contracted reduced expression for w', and that $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{N(w)-1}$ is a braid sequence for $D(\mathbf{i})$ with $N(w')$ terms. The conclusion follows. $\square$

3.2 Freely braided elements and the N-statistic.

The following lemma describes what happens when one goes up in the weak Bruhat order from a freely braided element.

Lemma 3.2.1. *Suppose that $w \in W$ is freely braided, and that $\ell(ws_i) > \ell(w)$ for some $i \in I$. As usual, we denote the simple root corresponding to i by α_i.*

(a) *Assume that α_i does not lie in any contractible inversion triple of ws_i.*

(i) *The roots occurring before α_i in any root sequence for ws_i are orthogonal to α_i.*

(ii) *The contractible inversion triples of ws_i are precisely those of the form $s_i(T)$, where T is a contractible inversion triple of w.*

(iii) *The element ws_i is freely braided and $N(ws_i) = N(w)$.*

(b) *Assume that α_i lies in some contractible inversion triple of ws_i.*

(i) *There is a reduced expression $\mathbf{i}$ for w of the form $\mathbf{u}ij\mathbf{v}$, where $j \in I$ does not commute with i and where each letter in $\mathbf{v}$ commutes with i.*

(ii) *Any reduced expression for w that is commutation equivalent to the reduced expression $\mathbf{i}$ in* (i) *must be of the form $\mathbf{u}'i\mathbf{v}_1j\mathbf{v}_2$, where each letter in $\mathbf{v}_1$ and each letter in $\mathbf{v}_2$ commutes with i.*

Note. We do not require above that ws_i be freely braided.

Proof. We first prove (a).

Since there is a reduced expression for ws_i in which s_i appears last, there is a root sequence $\bar{r}$ for ws_i in which α_i is the first root. Let $\bar{r}'$ be an arbitrary root sequence for ws_i. By the discussion following Proposition 1.2.1, $\bar{r}'$ may be obtained from $\bar{r}$ by applying a sequence of braid moves. Since none of these braid moves can be a long braid move involving α_i, we find that all the roots occurring before α_i in $\bar{r}'$ are orthogonal to α_i. This proves (i).

Suppose that T is a contractible inversion triple of w, and let $\bar{r}_0$ be a root sequence for w in which the elements of T appear consecutively. Since $(\alpha_i, s_i(\bar{r}_0))$ is a root sequence for ws_i (recall the definition of root sequence in §1.1), it follows that $s_i(T)$ is a contractible inversion triple of ws_i.

Conversely, suppose that T' is a contractible inversion triple of ws_i, and let $\bar{r}_1$ be a root sequence for ws_i in which the elements of T' appear consecutively. By hypothesis, α_i does not appear in T', and by (i), the elements appearing before α_i in $\bar{r}_1$ are orthogonal to α_i. Hence, we may apply short braid moves if necessary to obtain a root sequence $\bar{r}_1'$ for ws_i in which α_i appears first and in which the elements of T' still appear consecutively. Now, $\bar{r}_1'$ is of the form $(\alpha_i, s_i(\bar{r}_1''))$, where $\bar{r}_1''$ is a root sequence for w in which the elements of $s_i(T')$ occur consecutively. This proves (ii).

By (ii), we have $N(ws_i) = N(w)$. Let $\bar{r}_2$ be a root sequence for w of the form specified in [**5**, Theorem 4.2.3]. Using (ii) again, we see that the contractible inversion triples of ws_i, the roots in each of which appear consecutively in the root sequence $(\alpha_i, s_i(\bar{r}_2))$, are pairwise disjoint. Hence, ws_i is freely braided, and (iii) is proved.

We turn to (b).

Let $\bar{r}$ be a root sequence for ws_i in which α_i appears first, and consider a sequence of braid moves of minimal length subject to the condition that applying the sequence to $\bar{r}$ results in a root sequence $\bar{r}'$ in which the elements of some contractible inversion triple containing α_i appear consecutively. Denote by T the contractible inversion triple that contains α_i and is consecutive in $\bar{r}'$. By the minimality assumption, none of the braid moves in the above sequence is a long braid move involving α_i. Hence, every root occurring before α_i in $\bar{r}'$ is orthogonal to α_i, and we may therefore apply a sequence of short braid moves to $\bar{r}'$ to obtain a root sequence $\bar{r}''$ in which α_i appears first and in which the other elements of T appear consecutively. By Proposition 1.2.1, the sequence $\bar{r}''$ corresponds to a reduced expression for ws_i of the form $\mathbf{u}ij\mathbf{v}i$, where all of the letters in $\mathbf{v}$ commute with i. Deleting the rightmost i, we obtain a reduced expression for w of the required form, thus proving (i).

To prove (ii), we note that the reduced expression obtained in (i) is of the stated form, taking $\mathbf{u}' = \mathbf{u}$, $\mathbf{v}_1 = \emptyset$ and $\mathbf{v}_2 = \mathbf{v}$. The result follows, once we observe that applying a short braid move to a reduced expression of the form given in (ii) produces another expression of the same form. $\square$

The next result describes what happens when one goes down in the weak Bruhat order from a freely braided element.

Lemma 3.2.2. *Suppose that $w \in W$ is freely braided and that $i \in I$ satisfies $\ell(ws_i) < \ell(w)$. Then ws_i is freely braided, and we have*

$$N(ws_i) = \begin{cases} N(w) - 1 & \text{if } \alpha_i \text{ lies in a contractible inversion triple of } w, \\ N(w) & \text{otherwise.} \end{cases}$$

Proof. If $\bar{r}$ is a root sequence for ws_i, then $(\alpha_i, s_i(\bar{r}))$ is a root sequence for w. Therefore, every contractible inversion triple T of ws_i yields a contractible inversion triple $s_i(T)$ of w. This gives $N(w) \geq N(ws_i)$. The inequality is strict if α_i lies in a contractible inversion triple of w.

By the Exchange Condition for Coxeter groups (see [**6**, §5.8]), w has a reduced expression $\mathbf{i}$ ending with i, and by Proposition 3.1.3 (ii), there is a contracted reduced expression $\mathbf{j}$ for w that is commutation equivalent to $\mathbf{i}$. Write $\mathbf{j} = \mathbf{v}_1 i \mathbf{v}_2$, where each letter in $\mathbf{v}_2$ commutes with i. Suppose that α_i lies in a contractible inversion triple of w. Then $N(ws_i) \leq N(w) - 1$ by the first paragraph. On the other hand, it is clear that $\mathbf{v}_1\mathbf{v}_2$, a reduced expression for ws_i, has a braid sequence with $N(w) - 1$ terms; hence, $N(ws_i) \geq N(w) - 1$ by Proposition 1.2.1 (b)(i). Further, by Remark 3.1.4, ws_i has at least $2^{N(ws_i)}$ commutation classes. It now follows from Corollary 2.2.2 that ws_i is freely braided.

Suppose instead that α_i does not belong to a contractible inversion triple of w. Then, by Proposition 1.2.1 (b)(i), $\mathbf{v}_1\mathbf{v}_2$ has a braid sequence with $N(w)$ terms. It follows by the same proposition together with the first paragraph that $N(ws_i) = N(w)$. As above, we find that ws_i has at least $2^{N(ws_i)}$ commutation classes, and so is freely braided. $\square$

Remark 3.2.3. If $w \in W$ is freely braided and $\ell(ws_i) > \ell(w)$ with ws_i non-freely-braided, it may happen that $N(ws_i) > N(w) + 1$. For example, if W is of type A_3 and $w = s_2s_1s_3s_2$ (using the obvious indexing), then $N(w) = 0$ but $N(ws_3) = 2$.

3.3 Groups with finitely many freely braided elements.

The following lemma is a crucial ingredient in the proof of Theorem 3.3.3.

Lemma 3.3.1. *Let* $\mathbf{i}$ *be a contracted reduced expression for a freely braided element* $w \in W$ *with* $N(w) > 0$. *Then* $D(\mathbf{i})$ *is a contracted reduced expression for a freely braided element* w' *with* $N(w') = N(w) - 1$.

Proof. We start by writing

$$\mathbf{i} = \mathbf{c}_1\mathbf{b}_1\mathbf{c}_2\mathbf{b}_2 \cdots \mathbf{c}_{N(w)}\mathbf{b}_{N(w)}\mathbf{q},$$

where $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{N(w)}$ is the unique braid sequence for $\mathbf{i}$ with $N(w)$ terms. Let $\mathbf{i}_q$ be the expression obtained from $\mathbf{i}$ by deleting from $\mathbf{q}$ all but its first q letters. We thus have $\mathbf{i} = \mathbf{i}_l$, where l is the length of $\mathbf{q}$. If $\bar{r}$ is the root sequence of $\mathbf{i}$, then the first l roots in $\bar{r}$ are not involved in any contractible inversion triple of w, and it follows from repeated applications of Lemma 3.2.2 that for each q, the reduced expression $\mathbf{i}_q$ represents a freely braided element y_q with $N(y_q) = N(w)$. Moreover, if $q > 0$, then the first root in the root sequence of $\mathbf{i}_q$ does not lie in any contractible inversion triple of y_q.

We will prove by induction on q that $D(\mathbf{i}_q)$ is a reduced expression for a freely braided element w_q with $N(w_q) = N(w) - 1$. (These properties imply that $D(\mathbf{i}_q)$ is contracted.) Denote the letter that is deleted from $\mathbf{i}$ to form $D(\mathbf{i})$ by j.

Base case: $q = 0$**.**

In this case, $D(\mathbf{i}_0)$ is obtained from $\mathbf{i}_0$ by removing the last letter, j. It is clear that $D(\mathbf{i}_0)$ is a reduced expression for some group element w_0, which is freely braided by Lemma 3.2.2. The first three roots in the root sequence of $\mathbf{i}_0$ comprise a contractible inversion triple containing α_j. Hence, by Lemma 3.2.2 again, $N(w_0) = N(y_0) - 1$, and the latter equals $N(w) - 1$ by the first paragraph.

Inductive step: proof that $D(\mathbf{i}_q)$ is reduced.

Suppose that the statement is true for all q with $0 \leq q \leq k < l$. Let $q = k + 1$, and let i be the $(k+1)$-st factor of $\mathbf{q}$. By the inductive hypothesis, $D(\mathbf{i}_k)$ is a contracted reduced expression for a freely braided element w_k with $N(w_k) = N(w) - 1$. Assume toward a contradiction that $D(\mathbf{i}_{k+1})$ is not reduced. Then by the Exchange Condition for Coxeter groups, there is a reduced expression $\mathbf{i}'$ for w_k ending in i. According to Proposition 3.1.3 (ii), there is a unique contracted reduced expression $\mathbf{i}''$ for w_k that is both close to $D(\mathbf{i}_k)$ and commutation equivalent to $\mathbf{i}'$. We may write

$$\mathbf{i}'' = \mathbf{p}'' i \mathbf{c}'',$$

where all of the letters in $\mathbf{c}''$ commute with i.

By Lemma 3.1.6, $\mathbf{i}''$ must be of the form $D(\mathbf{j})$, where $\mathbf{j}$ is close to $\mathbf{i}_k$. The expression $\mathbf{j}$ is thus obtainable from $\mathbf{i}''$ by inserting the letter j at some point into the word. Since $\mathbf{i}_{k+1}$ is reduced, this insertion must take place to the right of the indicated occurrence of i in $\mathbf{i}''$. Therefore, $\mathbf{j}i$, which is a reduced expression for y_{k+1}, is of the form

$$\mathbf{p}'' i \mathbf{c}''_1 j \mathbf{c}''_2 i,$$

where each letter in $\mathbf{c}''_1$ and each letter in $\mathbf{c}''_2$ commutes with i. Applying short braid moves if necessary, we obtain

$$\mathbf{p}'' \mathbf{c}''_1 iji \mathbf{c}''_2,$$

and it follows from Proposition 1.2.1 (b) that the first root in the root sequence of $\mathbf{j}i$ belongs to a contractible inversion triple. Now, $\mathbf{j}$ (respectively, $\mathbf{j}i$) is a reduced expression for y_k (respectively, y_{k+1}), and $N(y_k) = N(y_{k+1}) = N(w)$. This contradicts Lemma 3.2.2, taking $w = y_{k+1}$.

We conclude that $D(\mathbf{i}_{k+1})$ is reduced.

Inductive step: proof that w_q is freely braided and $N(w_q) = N(w) - 1$.

By the above, we have $w_{k+1} = w_k s_i$ with $\ell(w_k s_i) > \ell(w_k)$. If α_i does not lie in any contractible inversion triple of $w_k s_i$, then the inductive step follows from Lemma 3.2.1 (a)(iii) (with w_k playing the role of w). Assume instead that we are in case (b) of Lemma 3.2.1, which is the only alternative.

By Lemma 3.2.1 (b)(i), the element w_k has a reduced expression of the form $\mathbf{u}ii'\mathbf{v}$, where $i' \in I$ does not commute with i and every letter in $\mathbf{v}$ commutes with i. Recall that $D(\mathbf{i}_k)$ is contracted by the inductive hypothesis. Hence, by Proposition 3.1.3 (ii), there is a contracted reduced expression $\mathbf{d}_k$ for w_k that is both close to $D(\mathbf{i}_k)$ and commutation equivalent to $\mathbf{u}ii'\mathbf{v}$. According to Lemma 3.2.1 (b)(ii), we have $\mathbf{d}_k = \mathbf{u}'i\mathbf{v}_1 i'\mathbf{v}_2$, where every letter in $\mathbf{v}_1$ and every letter in $\mathbf{v}_2$ commutes with i. Because $\mathbf{d}_k$ is close to $D(\mathbf{i}_k)$ and $N(w_k) = N(y_k) - 1$, Lemma 3.1.6 implies that $\mathbf{d}_k$ is of the form $D(\mathbf{i}'_k)$, where $\mathbf{i}'_k$ is a reduced expression for y_k that is close to $\mathbf{i}_k$ by a sequence of braid relations not involving $\mathbf{b}_{N(w)}$. There is no loss in generality in assuming that $\mathbf{i}_k$ is equal to $\mathbf{i}'_k$, so we will do this in order to make the arguments clearer.

Let $\mathbf{d}_{k+1} = \mathbf{d}_k i = \mathbf{u}'i\mathbf{v}_1 i'\mathbf{v}_2 i$. Since $\mathbf{d}_k = D(\mathbf{i}_k)$, we have $\mathbf{d}_{k+1} = D(\mathbf{i}_{k+1})$. Hence, by appropriately inserting j in $\mathbf{d}_{k+1}$, we obtain $\mathbf{i}_{k+1}$. The insertion must take place immediately to the right of a subword of $\mathbf{d}_{k+1}$ of the form jj', where $j' \in I$ does not commute with j.

We know from the first paragraph of the proof that α_i does not lie in a contractible inversion triple of y_{k+1}. The only way this can happen is if the letter j is inserted in $\mathbf{d}_{k+1}$ somewhere between the two indicated occurrences of i, and if j does not commute with i. Since the letter sitting two places to the left of the insertion site is also an occurrence of j, the latter occurrence of j must either occur to the left of the leftmost indicated occurrence of i in $\mathbf{d}_{k+1}$, or must be the indicated occurrence of i' in $\mathbf{d}_{k+1}$. We consider each of these two cases in turn.

In the first case, $\mathbf{i}_{k+1}$ can be written as

$$\mathbf{u}''jij\mathbf{v}_1 i'\mathbf{v}_2 i,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are as before and $\mathbf{u}''j = \mathbf{u}'$. Applying a long braid move, we obtain the following reduced expression for y_{k+1}:

$$\mathbf{u}''iji\mathbf{v}_1 i'\mathbf{v}_2 i.$$

This contradicts the fact (mentioned in the first paragraph of the proof) that y_{k+1} is freely braided, because Proposition 1.2.1 can now be used to show that the middle of the three indicated occurrences of i corresponds to a root that lies in two different contractible inversion triples.

In the second case, $i' = j$ and $\mathbf{i}_{k+1}$ can be written as

$$\mathbf{u}'i\mathbf{v}_1 jj'j\mathbf{v}_2' i,$$

where $\mathbf{v}_1$ is as above and $j'\mathbf{v}_2' = \mathbf{v}_2$, meaning that i commutes with j'. Applying a long braid move and commutations, we obtain

$$\mathbf{u}'\mathbf{v}_1 ij'jj'i\mathbf{v}_2'.$$

This again leads to a contradiction because the indicated occurrence of j corresponds to a root that belongs to two different contractible inversion triples of y_{k+1}.

We have completed the inductive step by showing that case (b) of Lemma 3.2.1 cannot occur. $\square$

Corollary 3.3.2. *Let* $\mathbf{i}$ *be a contracted reduced expression for a freely braided element* $w \in W$*, and maintain the notation of Definition 3.1.5, so that*

$$\mathbf{i} = \mathbf{u}_1\mathbf{b}_1\mathbf{u}_2\mathbf{b}_2 \cdots \mathbf{u}_{N(w)}\mathbf{b}_{N(w)}\mathbf{q}.$$

Then the expression obtained by omitting the rightmost letter in each of the words $\mathbf{b}_l$ *is a reduced expression for a fully commutative element.*

Proof. By applying Lemma 3.3.1 $N(w)$ times, we find that $D^{N(w)}(\mathbf{i})$, which is the expression described in the conclusion, is a reduced expression for a freely braided element w' with $N(w') = 0$. By Proposition 1.2.2, w' is fully commutative. □

Theorem 3.3.3. *A simply laced Coxeter group W has finitely many freely braided elements if and only if it has finitely many fully commutative elements.*

Proof. By Proposition 1.2.2, any fully commutative $w \in W$ satisfies $N(w) = 0$, and so is freely braided for vacuous reasons. This proves the "only if" part of the theorem.

Conversely, suppose that W has finitely many fully commutative elements. By Corollary 3.3.2, there is a map from the set of contracted reduced expressions for freely braided elements of W to the set of reduced expressions for fully commutative elements of W, given by

$$\mathbf{i} \mapsto D^{N(\phi(\mathbf{i}))}(\mathbf{i}).$$

Any element in the fibre over $D^{N(\phi(\mathbf{i}))}(\mathbf{i})$ can be recovered from $D^{N(\phi(\mathbf{i}))}(\mathbf{i})$ by making appropriate insertions of generators after certain subwords of the form ij, where $i, j \in I$ are noncommuting. Hence, the fibres of the above map are all finite. Since there are finitely many fully commutative elements and each of these has finitely many reduced expressions, there are finitely many fibres. It follows that W has finitely many freely braided elements. □

Independently of one another, Graham [**4**] and Stembridge [**8**] have classified the Coxeter groups with finitely many fully commutative elements. The classification has also been worked out in the simply laced case by Fan [**2**]. It turns out that a simply laced Coxeter group has finitely many fully commutative elements if and only if each connected component of its Coxeter graph is of type A_n, D_n or E_n for arbitrary n. In particular, Coxeter groups of type E_n for $n > 8$ have finitely many fully commutative elements, although the groups are infinite. This classification carries over for freely braided elements, by the above theorem.

References

[**1**] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres IV–VI*, Masson, Paris, 1981.

[**2**] C.K. Fan, *A Hecke algebra quotient and properties of commutative elements of a Weyl group*, Ph.D. thesis, M.I.T., 1995.

[**3**] C.K. Fan and J.R. Stembridge, *Nilpotent orbits and commutative elements*, J. Algebra **196** (1997), 490–498.

[**4**] J.J. Graham, *Modular representations of Hecke algebras and related algebras*, Ph.D. thesis, University of Sydney, 1995.

[**5**] R.M. Green and J. Losonczy, *Freely braided elements in Coxeter groups*, Ann. Comb. **6** (2002), 337–348.

[**6**] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.

[**7**] H. Matsumoto, *Générateurs et relations des groupes de Weyl généralisés*, C. R. Acad. Sci. Paris **258** (1964), 3419–3422.

[**8**] J.R. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), 353–385.

[**9**] J. Tits, *Le problème des mots dans les groupes de Coxeter*, Ist. Naz. Alta Mat. (1968), *Sympos. Math.*, vol. 1, Academic Press, London, 1969, pp. 175–185.