MATH 8174: Assignment 12

1. Let L be a finite-dimensional nilpotent Lie algebra over $\mathbb{C}$, let V be a finite-dimensional L-module, and let ϕ be a weight of V. Show that the weight space V_ϕ defined in (6.3) coincides with the subspace

$$U_\phi := \{v \in V : (x - \phi(x))^r v = 0 \text{ for some } r \text{ not depending on } x\},$$

and is therefore uniquely determined.

2. Prove that the normalizer $I_L(A)$ of a subalgebra A of L is itself a subalgebra, and is the largest subalgebra that contains A as an ideal.

3. Let $L = \mathfrak{gl}_n(\mathbb{C})$. Show that a diagonal matrix in L with distinct diagonal entries is a regular element of L. Deduce that the n-dimensional abelian subalgebra of all diagonal matrices is a Cartan subalgebra of L.

Here are some suggested steps.

(1) Show that if x is an upper triangular matrix in L with all diagonal entries equal to λ, then $\dim(L_{0,x}) \geq \frac{1}{2}n(n+1)$.

(2) Let $x \in L$. Denote the distinct eigenvalues of x by $\lambda_1, \lambda_2, \ldots$ and let $m_1, m_2, \ldots$ be their respective multiplicities. Let y be a Jordan canonical form of x. Show that the space $L_{0,x} = L_{0,y}$ has dimension at least equal to $\sum_i \frac{1}{2}m_i(m_i+1)$, and that this latter number is strictly greater than n if the eigenvalues are not distinct.

(3) If the eigenvalues of $x \in L$ are all distinct, show that $L_{0,x}$ is conjugate to the subalgebra H of all diagonal matrices. (Notice that this implies that all Cartan subalgebras are conjugate. This happens more generally.)

4. Show that over any field k, there exists a 3-dimensional Lie algebra L with basis $\{a, b, c\}$ satisfying

$$[a, b] = c, \quad [b, c] = a, \text{ and } \quad [c, a] = b.$$

Write down the matrix for $\operatorname{ad}(\lambda a+\mu b+\nu c)$ with respect to this basis and compute its characteristic polynomial. Deduce that $\lambda a + \mu b + \nu c$ is regular if and only if $\lambda^2 + \mu^2 + \nu^2 \neq 0$. Conclude that $\langle a \rangle$ is one of the Cartan subalgebras of L, and find its weights and the corresponding weight spaces when $k = \mathbb{C}$.