MATH 8174: Assignment 8

1. Show that the three matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis for the Lie algebra $\mathfrak{sl}_2(k)$. Work out the 3^3 structure constants with respect to this basis.

2. Let X, Y and Z be subspaces of the Lie algebra L. Show that:

(a) $[X, Y] = [Y, X]$;

(b) X is a subalgebra if and only if $X^2 \subseteq X$;

(c) $[[X, Y], Z] \subseteq [[Y, Z], X] + [[Z, X], Y]$. Also,

(d) show that any bracketing of an n-fold product $[\cdots [[[X, X], X], X], \cdots X]$ is contained in X^n.

3. Let $\theta : L \to M$ be a homomorphism of Lie algebras. Show that $[L, \ker(\theta)] \subseteq \ker(\theta)$.

4. Let $n \geq 2$ and let S denote the set of all $n \times n$ skew-symmetric matrices over k, where k is a field of characteristic different from 2. Show that S is a subalgebra of $\mathfrak{gl}_n(k)$, but not of $M_n(k)$.

5. Let J and K be ideals of L. Show that $[J, K]$, $J + K$ and $J \cap K$ are also ideals of L. Show that the center $Z(L)$ of L is an ideal of L.

6. Let L be a 2-dimensional Lie algebra over k, and suppose that $L^2 \neq 0$. Show that L is isomorphic to $\mathfrak{sl}_2(k) \cap \mathfrak{t}_2(k)$, where $\mathfrak{t}_2(k)$ is the Lie algebra of upper triangular 2×2 matrices over k. What happens in characteristic 2?

7. Let J and K be ideals of L with $J \leq K$. Prove that $K/J \trianglelefteq L/J$ and that $(L/J)/(K/J) \cong L/K$.

8. Let L be a nonabelian 2-dimensional Lie algebra. Show the center of L, $Z(L)$, is trivial.