MATH 8174: Assignment 5

1. Let χ be the character of a complex representation of a finite group G. Show that the set of $g \in G$ with $\chi(g) = \chi(1)$ is a normal subgroup N of G. It is called the *kernel* of the character χ. Show that χ can be regarded as the character of a representation of G/N. Conversely, show that every representation of G/N gives rise to a representation of G in this way.

2. Suppose that H is a subgroup of a finite group G. Show that each irreducible complex representation of G is contained in a representation induced from an irreducible representation of G. (Hint: induce up the regular representation of H.) Deduce that if A is an abelian subgroup of G then the dimension of each irreducible representation of G is at most $|G : A|$.

3. Show using characters that if V is a complex representation of a finite group G, and W is a representation of a subgroup H of G, then there is an isomorphism

$$V \otimes (W \uparrow^G) \cong (V \downarrow_H \otimes W) \uparrow^G$$

of $\mathbb{C}G$-modules.

4. A *generalized character* of a finite group G is a class function that can be expressed as the difference of two characters. Show that the set of generalized characters of G forms a subring of the ring of class functions on G (the ring operations on class functions are pointwise addition and multiplication) and that a class function ϕ is a generalized character if and only if for each irreducible character χ of G we have $\langle \phi, \chi \rangle \in \mathbb{Z}$, where

$$\langle \phi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)} \chi(g).$$

Show that a generalized character ϕ is the character of an irreducible representation if and only if $\langle \phi, \phi \rangle = 1$ and $\phi(1) > 0$. Show that Frobenius reciprocity holds for generalized characters, for a suitable definition of induction on generalized characters.