MATH 6140: Final examination. Tuesday, 9 May 2023.

Put **your name** on each answer sheet. Answer the first **three** questions; the fourth question is optional.

Justify your answers in full. Formula sheets, calculators, notes and books are not permitted.

1. Let $\sqrt[5]{2}$ denote the real fifth root of 2, let $\zeta = e^{2\pi i/5}$ be a primitive fifth root of unity, and define

$$\alpha_1 = \sqrt[5]{2}, \quad \alpha_2 = \sqrt[5]{2}\zeta, \quad \alpha_3 = \sqrt[5]{2}\zeta^2, \quad \alpha_4 = \sqrt[5]{2}\zeta^3, \quad \text{and} \quad \alpha_5 = \sqrt[5]{2}\zeta^4.$$

(i) Prove that $K = \mathbb{Q}(\sqrt[5]{2}, \zeta)$ is the splitting field over $\mathbb{Q}$ of the polynomial $x^5 - 2$, and show that the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ has order 20.

(ii) Prove that there exist unique elements $\sigma, \tau \in \mathrm{Gal}(K/\mathbb{Q})$ satisfying

$$\begin{array}{rcl} \sigma(\sqrt[5]{2}) & = & \sqrt[5]{2}\zeta \\ \sigma(\zeta) & = & \zeta \end{array} \quad \text{and} \quad \begin{array}{rcl} \tau(\sqrt[5]{2}) & = & \sqrt[5]{2} \\ \tau(\zeta) & = & \zeta^2 \end{array}.$$

(iii) Prove that, as permutations of the roots $\alpha_1, \ldots, \alpha_5$, the automorphisms σ and τ correspond to the elements $(1\,2\,3\,4\,5)$ and $(2\,3\,5\,4)$ of the symmetric group S_5.

(iv) Deduce that σ and τ generate $\mathrm{Gal}(K/\mathbb{Q})$, and that $\mathrm{Gal}(K/\mathbb{Q})$ is a nonabelian group whose Sylow p-subgroups are all cyclic.

[**Hint**: If you get stuck, you may find it helpful to list all 20 elements of the Galois group as permutations, although this is not necessary.]

2. Maintain the notation of Question 1.

(i) Show that there are precisely five subfields of K (say, F_1, F_2, F_3, F_4, and F_5) that have degree 5 over K. Express each one in the form $F_i = \mathbb{Q}(\beta_i)$, where the β_i are elements you should find explicitly.

(ii) Show that the fields $F_1, \ldots, F_5$ are all isomorphic to each other as rings, and that none of them is Galois over $\mathbb{Q}$.

(iii) For each of the fields F_i in (ii), show that there exists a unique subfield E_i of K such that $F_i \leq E_i$ and $[E_i : \mathbb{Q}] = 10$.

(iv) Show that the subfields E_i are the only subfields of K that have degree 10 over $\mathbb{Q}$.

(v) Show that the subfields E_i are all isomorphic to each other as rings, and that none of them is Galois over $\mathbb{Q}$.

[continued overleaf]

3. Maintain the notation of Questions 1 and 2.

(i) Explain why complex conjugation restricts to an element $\rho \in \mathrm{Gal}(K/\mathbb{Q})$. Express ρ as a permutation of the roots. [“As a permutation of the roots” means “either as an element of S_5, or as a word in the generators σ and τ and their inverses”.]

(ii) Let L be the subfield of K given by $L = \mathbb{Q}(\zeta)$. Prove that $L/\mathbb{Q}$ is Galois, and that L is the unique subfield of K that is of degree 4 over $\mathbb{Q}$. [“Unique” means “unique as a subset of K”, not “unique up to isomorphism”.]

(iii) Show that $L' = \mathbb{Q}(\zeta + \zeta^{-1})$ is the unique subfield of K that has degree 2 over $\mathbb{Q}$, and determine whether $L'/\mathbb{Q}$ is Galois.

(iv) Show that $E' = \mathbb{Q}(\sqrt[5]{2},\ \zeta + \zeta^{-1})$ is one of the subfields E_i. Find the subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ that corresponds to E', as an explicit set of permutations.

(v) Show that K has precisely 14 subfields, including K itself and $\mathbb{Q}$, and that precisely four of these are Galois over $\mathbb{Q}$.

4. [For **20** bonus points up to a maximum of 200.]

(i) Show that $\mathbb{Q}(\sqrt{5})$ is a subfield of the field K of Question 1.

(ii) Express the group $H = \mathrm{Gal}(K/\mathbb{Q}(\sqrt{5}))$ as an explicit group of permutations of the roots of $x^5 - 2$. [It is sufficient to give a set of permutations that generate H.]

(iii) To which familiar group is H isomorphic?