

**MATH 6140: Second midterm examination. Wednesday, 22 March 2023.**

Put **your name** on each answer sheet. Answer **both** questions.

*Justify all your answers in full.*

*Formula sheets, calculators, notes and books are not permitted.*

1. Let  $F$  be a field, let  $M_n(F)$  be the ring of  $n \times n$  matrices with coefficients in  $F$ , and let  $I \in M_n(F)$  be the identity matrix. A matrix  $A \in M_n(F)$  is said to be *unipotent* if  $(A - I)^k = 0$  for some integer  $k > 0$ ; in other words,  $A$  is unipotent if and only if  $A - I$  is nilpotent.

Show that there are five similarity classes of unipotent matrices in  $M_4(F)$ . For each similarity class, find the associated (a) elementary divisors, (b) invariant factors, (c) minimal polynomial, (d) characteristic polynomial, (e) Rational Canonical Form, and (f) Jordan Canonical Form.

2. Maintain the notation of Question 1.
  - (i) Find the determinant of a unipotent matrix in  $M_n(F)$ .
  - (ii) Give an example of a unipotent matrix whose trace is zero.
  - (iii) Show that any unipotent matrix in  $M_n(\mathbb{C})$  is similar (over  $\mathbb{C}$ ) to a matrix in  $M_n(\mathbb{Q})$ .
  - (iv) Classify the diagonalizable unipotent matrices in  $M_n(F)$ .