MATH 6140: First midterm examination. Wednesday, 15 February 2023.

Put **your name** on each answer sheet. Answer **all three** questions.

Justify all your answers in full.

Formula sheets, calculators, notes and books are not permitted.

1. Let $V = \mathbb{R}^2$ be a 2-dimensional real vector space considered as a unital R-module, where $R = \mathbb{R}[x]$, and where x acts by the linear transformation $T : V \to V$ whose matrix with respect to the standard basis $\mathcal{E} = \{e_1, e_2\}$ is given by
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
(Note that, geometrically, T is rotation counterclockwise about the origin by a right angle.) Prove that the endomorphism ring $\mathrm{End}_{\mathbb{R}[x]}(V)$ is a field.

2. Let F be a field, and let $R = F[x]$ be the ring of polynomials over F in the indeterminate x. Let $F[x^3]$ be the ring of polynomials over F in x^3; in other words, $F[x^3]$ is the subring of $F[x]$ consisting of all polynomials of the form $\sum_{i=0}^{\infty} a_i x^{3i}$. You may assume without proof that S is a subring of R, and that R becomes a left S-module under multiplication $s.r = sr$. Prove that, as left S-modules, we have
$$R \cong S \oplus S \oplus S;$$
in other words, that R is a free S-module of rank 3.

3. Let $\mathbb{F}_2$ be a field with two elements. The group $GL_2(\mathbb{F}_2)$ is isomorphic to a well-known group of small order; determine which one.