MATH 6140: Final examination. Saturday, 1 May 2021.

Put **your name** on each answer sheet. Answer the first **three** questions; the fourth question is optional.

Justify your answers in full. Formula sheets, notes and books are not permitted.

1. Throughout this exam, let $\zeta = e^{2\pi i/24}$ be a primitive 24th root of unity, and let $K = \mathbb{Q}(\zeta)$ be the associated cyclotomic field.

(i) Explain why $K/\mathbb{Q}$ is a Galois extension of degree 8.

(ii) Show that the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ is given by the set

$$\{\sigma_a : a \in \{1, 5, 7, 11, 13, 17, 19, 23\}\},$$

where $\sigma_a(\zeta) = \zeta^a$.

(iii) The restriction of complex conjugation to K is an element of G. Which of the elements σ_a does complex conjugation restrict to?

(iv) Without using answers to later questions, verify that every $g \in G$ satisfies $g^2 = e$, where $e = \sigma_1$ is the identity automorphism.

2. Let ζ and K be as in Question 1.

(i) Show that $\zeta^2 + \zeta^{-2} = \sqrt{3}$ and that $\zeta^3 + \zeta^{-3} = \sqrt{2}$. Show that we also have $i \in K$.

(ii) Prove that K is equal (as a subset of $\mathbb{C}$) to $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$. [You may use without proof the standard result that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ has degree 4 over $\mathbb{Q}$.]

(iii) Prove that we have

$$K \cap \mathbb{R} = \mathbb{Q}(\zeta + \zeta^{-1}) = \mathbb{Q}(\sqrt{2}, \sqrt{3}).$$

[You may use any standard results about cyclotomic extensions without proof if you state them clearly.]

(iv) Prove that $\cos(\pi/12) \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(v) Can the regular 24-gon be constructed by straightedge and compass? Why or why not?

3. Maintain the previous notation.

(i) Show that the subfield $E_1 = \mathbb{Q}(\sqrt{2})$ corresponds under the Galois correspondence to the subgroup $H_1 = \{\sigma_a : a \in \{1, 7, 17, 23\}\}$ of G.

(ii) Show that the subfield $E_2 = \mathbb{Q}(\sqrt{3})$ corresponds under the Galois correspondence to the subgroup $H_2 = \{\sigma_a : a \in \{1, 11, 13, 23\}\}$ of G.

(iii) Show that the subfield $E_3 = \mathbb{Q}(i)$ corresponds under the Galois correspondence to the subgroup $H_3 = \{\sigma_a : a \in \{1, 5, 13, 17\}\}$ of G.

(iv) Find the fixed field, F_1, of the subgroup $\{\sigma_1, \sigma_{17}\}$ of G, and express F_1 in the form $\mathbb{Q}(\zeta^k)$ for some suitable integer k.

(v) Find the fixed field, F_2, of the subgroup $\{\sigma_1, \sigma_{13}\}$ of G, and express F_2 in the form $\mathbb{Q}(\zeta^k)$ for some suitable integer k.

(vi) Find the fixed field, F_3, of the subgroup $\{\sigma_1, \sigma_{23}\}$ of G. Explain why F_3 cannot be expressed in the form $\mathbb{Q}(\zeta^k)$ for any integer k.

4. [For 10 bonus points up to a maximum of 200.]

Prove that

$$\cos(\pi/12) = \frac{\sqrt{2}}{2(\sqrt{3}-1)} = \frac{\sqrt{2}+\sqrt{6}}{4}.$$
