MATH 6140: First midterm examination. Friday, 19 February 2021.

Put **your name** on each answer sheet. Answer **all three** questions.

Justify all your answers in full.

Formula sheets, calculators, notes and books are not permitted.

1. Let $V = \mathbb{R}^2$ be a 2-dimensional real vector space considered as a unital R-module, where $R = \mathbb{R}[x]$, and where x acts by the linear transformation $T : V \to V$ whose matrix with respect to the standard basis $\mathcal{E} = \{e_1, e_2\}$ is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(i) Find all the R-submodules of V.

(ii) Is V a cyclic R-module? Why or why not?

2. Let V be the $\mathbb{R}[x]$-module of Question 1, equipped with the basis $\mathcal{E}$.

(i) Let $\phi : V \to V$ be an $\mathbb{R}$-linear map, and let A be the matrix of ϕ with respect to the basis $\mathcal{E}$. Find a necessary and sufficient condition on A for ϕ to be a homomorphism of $\mathbb{R}[x]$-modules.

(ii) Show that there exists a nonzero element $\psi \in \mathrm{End}_{\mathbb{R}[x]}(V)$ for which $\psi^2 = 0$, and deduce that $\mathrm{End}_{\mathbb{R}[x]}(V)$ is not a division ring.

3. Let D be an $n \times n$ diagonal matrix over a field F. Prove that

$$\det(D \otimes D) = \det(D)^{2n},$$

where $\otimes$ is the Kronecker product.