MATH 4140: Practice Final

1. The symmetric group S_{11} has a subgroup of order 7920 known as M_{11}. The character table of M_{11} is

M_{11}	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	10	2	1	2	0	-1	0	0	-1	-1
χ_3	10	-2	1	0	0	1	α	$-\alpha$	-1	-1
χ_4	10	-2	1	0	0	1	$-\alpha$	α	-1	-1
χ_5	11	3	2	-1	1	0	-1	-1	0	0
χ_6	16	0	-2	0	1	0	0	0	β	$\overline{\beta}$
χ_7	16	0	-2	0	1	0	0	0	$\overline{\beta}$	β
χ_8	44	4	-1	0	-1	1	0	0	0	0
χ_9	45	-3	0	1	0	0	-1	-1	1	1
χ_{10}	55	-1	1	-1	0	-1	1	1	0	0

where $\alpha = \sqrt{2}i$ and $\beta = (-1+\sqrt{11}i)/2$.

(i) Prove that M_{11} is a nonabelian simple group.

(ii) Because M_{11} is a subgroup of S_{11}, it follows that M_{11} has a permutation representation

$$\rho : M_{11} \to GL(11, \mathbb{C})$$

of degree 11. Find the character of the representation ρ.

(iii) The group M_{11} has a permutation representation

$$\rho' : M_{11} \to GL(12, \mathbb{C})$$

of degree 12 with the property that the element $g_4 \in M_{11}$ acts with no fixed points. Find the character of the representation ρ'.

(iv) Prove that if $g \in M_{11}$ is conjugate to g_7, g_8, g_9, or g_{10}, then g is not conjugate in M_{11} to g^{-1}.

(v) Show that $\chi_2\chi_2 = \chi_1 + \chi_2 + \chi_8 + \chi_9$.

(vi) Let $\chi_{A,2}$ be the exterior square of χ_2. Use your answer to (v) to show that we either have $\chi_{A,2} = \chi_1 + \chi_8$ or $\chi_{A,2} = \chi_9$.

(vii) Suppose that $x \in M_{11}$ has order 2. Using only the entries in the character table, show that x must be conjugate to either g_2 or g_4. [This can be done with very little computation.]

(viii) Use the formula for the character of an exterior square to show that the element x of order 2 satisfies $\chi_{A,2}(x) = -3$. Deduce that x is conjugate to g_2 and that $\chi_{A,2} = \chi_9$.

(ix) Show that χ_{10} is not the character of the symmetric square $\chi_{S,2}$ of χ_2.

[continued overleaf]

2. There is a certain group G of order 42 with seven conjugacy classes represented by $g_1, g_2, \ldots, g_7$. One of the irreducible characters of G is given below. The number $\zeta = e^{2\pi i/6}$ is a primitive 6-th root of unity.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
λ	1	ζ	ζ^2	-1	ζ^4	ζ^5	1

(i) Complete the character table of G.
(ii) Find the size of each conjugacy class of G.
(iii) Show that G has precisely five normal subgroups: the group itself, the trivial subgroup, and three others.
(iv) Identify which of these five normal subgroups is equal to (a) the center of G and (b) the commutator subgroup of G.
(v) Show that if N is a nontrivial normal subgroup of G, then G/N is abelian.

3. **Bonus/MATH 5140 question.** Let G be a group and let G' be the commutator subgroup of G. The group G is called *perfect* if $G' = G$. (For example, the alternating group A_5 is perfect.) Prove that there is no perfect group of order 24.