MATH 4140: Practice Midterm 2

A certain group G has conjugacy class representatives $g_1, g_2, \dots, g_6$ and character table

	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	7	-1	-1	1	0	0
χ_3	8	0	0	-1	1	1
χ_4	3	-1	1	0	α	$\overline{\alpha}$
χ_5	3	-1	1	0	$\overline{\alpha}$	α
χ_6	6	2	0	0	-1	-1

where $\alpha = (-1 + \sqrt{7}i)/2$ and $\overline{\alpha} = (-1 - \sqrt{7}i)/2$.

1.

(i) Show that G has order 168.

(ii) Find the sizes of each conjugacy class and the orders of the corresponding centralizers. [**Hint:** you should find that every conjugacy class has size 1, 21, 24, 42, or 56.]

(iii) Show that the center, $Z(G)$, of G is trivial.

(iv) Show that the conjugacy class g_6^G consists precisely of the inverses of the conjugacy class g_5^G.

(v) Let χ be a (not necessarily irreducible) character of G such that $\chi(g) \in \mathbb{Z}$ for all $g \in G$, and suppose that $\chi(1) \leq 5$. By considering the expression of χ as a sum of irreducible characters, prove that $\chi = k\chi_1$, where k is the integer $\chi(1)$. (In other words, prove that χ is the character of the trivial representation of degree k.)

(vi) Let $\rho : G \to GL(n, \mathbb{C})$ be a permutation representation of G; that is, a representation for which $\rho(g)$ is a permutation matrix for all $g \in G$. Using the result of (v), prove that if $n \leq 6$, then ρ is the trivial homomorphism.

2. **Bonus/MATH 5140 question.**

(i) Let χ be a character of G such that $\chi(g) \in \mathbb{Z}$ for all $g \in G$, and suppose that $\chi(1) = 6$. Prove that we must have either $\chi = 6\chi_1$, or $\chi = \chi_4 + \chi_5$, or $\chi = \chi_6$.

(ii) It turns out that G is isomorphic to a subgroup of the alternating group A_7, which means that there is a nontrivial representation $\rho : G \to GL(7, \mathbb{C})$ such that $\rho(g)$ is a permutation matrix for all $g \in G$. Assuming that this is true, prove that the character of ρ is $\chi_1 + \chi_6$.