MATH 4140: Practice Midterm

1. Let G be the dihedral group D_8, given by the usual presentation

$$G = \langle a, b : a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$

Show that there is a representation $\rho : G \to GL(3, \mathbb{C})$ satisfying

$$\rho(a) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

2. Let $V_\rho = \mathbb{C}^3$ be the module corresponding to the representation ρ of Question 1, and let E be the matrix

$$E = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

Let $\phi : V_\rho \to V_\rho$ be the linear map given by $\phi(v) = Ev$.

(i) Prove that ϕ is a homomorphism of $\mathbb{C}G$-modules.

(ii) Show that the following are submodules of V_ρ:

$$V_1 = \left\{ \begin{bmatrix} a \\ b \\ b-a \end{bmatrix} : a, b \in \mathbb{C} \right\} \quad \text{and} \quad V_2 = \left\{ \begin{bmatrix} c \\ 0 \\ c \end{bmatrix} : c \in \mathbb{C} \right\}.$$

Hint: consider the kernel and image of ϕ.

(iii) Show that the module V_2 does not afford the trivial representation of G.

3. **Bonus/MATH 5140 question:** List the elements in the kernel of the representation ρ_2 afforded by V_2. Express your answer as a subset of

$$\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$