MATH 4140: Assignment 13

Chapter 21

1. Let G be a finite group and let K be a subgroup of G. Let χ and ψ be the trivial characters of G and K, respectively.
   Use the Frobenius Reciprocity Theorem (Theorem 21.16) to show that $\psi{\uparrow}_K^G$ contains χ as a constituent with multiplicity 1; that is, $\langle \psi{\uparrow}_K^G, \chi\rangle = 1$.
   (The reason this is important is that $\psi{\uparrow}_K^G$ is the character of the permutation module arising from the action of G on the left cosets of K: $g.(xK) = gxK$. We already know that every permutation module contains a copy of the trivial module, but this shows that there is only one copy of the trivial module when the module comes from an action on cosets.)

2. Let G be a finite group, and let $H \leq G$. Let V be a $\mathbb{C}G$-module with character χ, and let let $W = \mathrm{Span}(\{w\})$ be the trivial $\mathbb{C}H$-module with character ψ. We say that a vector $v \in V$ is *H-invariant* if $h.v = v$ for all $h \in H$. You may assume without proof (but you should convince yourself) that v is H-invariant if and only if there exists a homomorphism $\phi : W \to V$ of $\mathbb{C}H$-modules such that $\phi(w) = v$. You may also assume without proof that V has a nonzero H-invariant vector if and only if

$$\langle \psi, \chi{\downarrow}_H^G\rangle \neq 0.$$

   Recall that the character table of the symmetric group $G = S_4$ is as follows; the sizes of the centralizers have also been included.

	1	(12)	(123)	(12)(34)	(1234)
	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

   The group G has an abelian subgroup $H = \{1, (12), (34), (12)(34)\}$, which is isomorphic to V_4, but which is not normal in G.

(i) Let ψ be the trivial character of H. Express the character $\psi{\uparrow}_H^G$ as a sum of irreducible characters of G. [Hint: Question 1 shows that the trivial character of G will be one of the characters that appears.]

(ii) Let V be an irreducible $\mathbb{C}G$-module. Show that V has a nonzero H-invariant vector if and only if the character of V is χ_1, χ_3 or χ_4.

[continued overleaf]

3. The group $G = S_5$ and two of its irreducible characters, with centralizer sizes, are as follows (see page 201 of the book).

	1	(12)	(123)	(12)(34)	(1234)	(123)(45)	(12345)
	120	12	6	8	4	6	5
χ_1	1	1	1	1	1	1	1
χ_7	5	-1	-1	1	1	-1	0

The group G has a subgroup K of order 20, generated by (12345) and (2354). The character table of K, together with centralizer sizes, is as follows (see also page 294 of the book).

	1	(12345)	(2354)	(25)(34)	(2453)
	20	5	4	4	4
ψ_1	1	1	1	1	1
ψ_2	1	1	i	-1	$-i$
ψ_3	1	1	-1	1	-1
ψ_4	1	1	$-i$	-1	i
ψ_5	4	-1	0	0	0

Let $\psi_1\!\uparrow_K^G$ be the character of G obtained by inducing the trivial character of K.

(i) Show that the values of $\psi_1\!\uparrow_K^G$ on the conjugacy classes of 1, (12), (123), and (123)(45) are 6, 0, 0, and 0, respectively. [This does not require any detailed calculations. Look at Corollary 21.20 and Proposition 21.23 (1) in the book.]

(ii) Use Proposition 21.23 (2) to calculate the values of $\psi_1\!\uparrow_K^G$ on the conjugacy classes (12)(34), (1234), and (12345).

(iii) Show that we have $\psi_1\!\uparrow_K^G = \chi_1 + \chi_7$, where χ_1 and χ_7 are the characters of S_5 defined above.

The significance of the result of (iii) is that, combined with the result of Question 1, it shows that χ_7 is the "fixed points minus 1" character of a permutation representation, namely the degree 6 representation arising from the action of G on the left cosets of its index 6 subgroup K. However, this action has some surprising features: the elements (12), (123), and (123)(45) are all acting with no fixed points!

[continued overleaf]

4. Let $n \geq 2$, let ψ be an irreducible character of the alternating group A_n, and let ε be the sign character of the symmetric group S_n.

(i) Use the results of Chapter 20 and Frobenius reciprocity to show that exactly one of the following situations (a) or (b) must occur:

(a) the character $\psi\uparrow_{A_n}^{S_n}$ is an irreducible character χ such that $\chi(1) = 2\psi(1)$ and $\varepsilon\chi = \chi$; or

(b) the character $\psi\uparrow_{A_n}^{S_n}$ is the sum of two distinct irreducible characters $\chi + \chi'$ such that $\psi(1) = \chi(1) = \chi'(1)$ and $\chi' = \varepsilon\chi$.

(ii) The degrees of the irreducible characters of S_5 are $1, 1, 4, 4, 5, 5, 6$, and the degrees of the irreducible characters of A_5 are $1, 3, 3, 4, 5$. For each irreducible character ψ of A_5, express $\psi\uparrow_{A_5}^{S_5}$ as a sum of the irreducible characters of S_5. [The information given here is enough; you do not need to look at the full character table.]