MATH 4140: Assignment 12

Chapter 20

1. The symmetric group S_5 has a subgroup G, called the Frobenius group of order 20. The character table of G is as follows.

	g_1	g_2	g_3	g_4	g_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	1	-1	i	$-i$	1
χ_4	1	-1	$-i$	i	1
χ_5	4	0	0	0	-1

(i) Show that the conjugacy classes of g_1, g_2, g_3, g_4 and g_4 have sizes $1, 5, 5, 5, 4$, respectively.
(ii) Show that $H = \ker(\chi_2)$ is an index 2 subgroup of G.
(iii) By multiplying the irreducible characters by χ_2, prove that
(a) both $\chi_1{\downarrow_H}$ and $\chi_2{\downarrow_H}$ are the trivial character, ψ_1, of H;
(b) both $\chi_3{\downarrow_H}$ and $\chi_4{\downarrow_H}$ are a nontrivial linear character, ψ_2 of H;
(c) $\chi_5{\downarrow_H} = \psi_3 + \psi_4$ is the sum of two distinct characters ψ_3 and ψ_4 of H, each of which has degree 2.
(iv) Prove that the conjugacy classes of g_1 and g_2 in G are also conjugacy classes in H, but that the conjugacy class of g_5 breaks into two conjugacy classes h_3 and h_4 of H, each of which has size 2.
(v) Prove that the character table of H has the form

	g_1	g_2	h_3	h_4
ψ_1	1	1	1	1
ψ_2	1	-1	1	1
ψ_3	2	β_1	β_2	β_3
ψ_4	2	β_4	β_5	β_6

(vi) Use the column orthogonality relations on the g_1 and g_2 columns to prove that $\beta_1 = \beta_4 = 0$.

[continued overleaf]

(vii) Use orthogonality relations between (a) columns 1 and 3; (b) columns 1 and 4; (c) rows 1 and 3; and (d) rows 1 and 4 to show that the character table of H has the form

	g_1	g_2	h_3	h_4
ψ_1	1	1	1	1
ψ_2	1	-1	1	1
ψ_3	2	0	α	$-1-\alpha$
ψ_4	2	0	$-1-\alpha$	α

(viii) Assuming that $\alpha \in \mathbb{R}$, complete the character table of H.

(ix) There are two groups of order 10 up to isomorphism: the cyclic group C_{10} and the dihedral group D_{10}. Which one of these is isomorphic to H?

2. The dihedral group D_{12} is given by the presentation

$$D_{12} = \langle a, b : a^6 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$$

and the cyclic group C_6 is given by the presentation

$$C_6 = \langle a : a^6 = 1 \rangle.$$

The character tables of the two groups are as follows, where $\omega = e^{2\pi i/6}$.

D_{12}	1	a^3	a	a^2	b	ab
χ_1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1
χ_3	1	-1	-1	1	1	-1
χ_4	1	-1	-1	1	-1	1
χ_5	2	-2	1	-1	0	0
χ_6	2	2	-1	-1	0	0

C_6	1	a	a^2	a^3	a^4	a^5
ψ_1	1	1	1	1	1	1
ψ_2	1	ω	ω^2	-1	ω^4	ω^5
ψ_3	1	ω^2	ω^4	1	ω^2	ω^4
ψ_4	1	-1	1	-1	1	-1
ψ_5	1	ω^4	ω^2	1	ω^4	ω^2
ψ_6	1	ω^5	ω^4	-1	ω^2	ω^1

(i) Show that the kernel of χ_2 is the subgroup $\langle a \rangle \leq D_{12}$, which consists of the rotations.

(ii) For each i with $1 \leq i \leq 6$, express the restricted character $\chi_i{\downarrow_{C_6}}$ as a linear combination of the ψ_j. (It follows from the theory of restricted characters that each restricted character will either be one of the ψ_j, or a sum of two distinct ψ_j.)

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3. Consider the character table of D_{12}, as given in Question 2.

(i) Show that the kernel of χ_3 is a subgroup H of order 6.

(ii) Show that every irreducible character χ of D_{12} has the property that $\chi{\downarrow_H}$ is irreducible.

(iii) Give the character table for H.

(iv) To which familiar group is H isomorphic?

(v) If we repeat this exercise with χ_4 in place of χ_3, does something similar happen, or is the situation more like that the one in Question 2? How can you tell?

(vi) Show that the normal subgroups of D_{12} are as follows:

- the whole group D_{12};
- three subgroups of order 6;
- the commutator subgroup, of order 3;
- the center, of order 2;
- the trivial subgroup.

(vii) How many of the three subgroups of order 6 are abelian?