MATH 4140: Assignment 11

Chapter 19

1. Let G be the alternating group A_5, let χ_1 be the trivial character of G, and let χ_2 be the irreducible character of G given by

	1	(123)	(12)(34)	(12345)	(13524)
χ_2	3	0	-1	α	β

where $\alpha = (\sqrt{5}+1)/2$ and $\beta = (-\sqrt{5}+1)/2$. The conjugacy class sizes, in order, are 1, 20, 15, 12, and 12.
This question will use the character χ_2 to construct the entire character table of G. Recall from the discussion about rotation angles in Assignment 8, Question 4 that if $x \in G$ is a 5-cycle, then x is never conjugate to x^2. The characters χ_S and χ_A appearing in Proposition 19.14 of the book are known respectively as the *symmetric square* and the *exterior square* of the character χ.

(i) Show that the symmetric square, $\chi_{2,S}$, of χ_2 is an integer-valued character of degree 6. Show that the trivial character χ_1 is a constituent of $\chi_{2,S}$ (as defined in Definition 14.19). Show that $\chi_5 := \chi_{2,S} - \chi_1$ is an irreducible character of degree 5.

(ii) Show that the symmetric square, $\chi_{5,S}$, of χ_5 is an integer-valued character of degree 15. Show that each of χ_5 and χ_1 is a constituent of $\chi_{5,S}$. Show that there is an integer-valued, degree 4 irreducible character χ_4 of the form $\chi_{5,S} - c\chi_5 - d\chi_1$ for suitable positive integers c and d.

(iii) Show that the exterior square, $\chi_{4.A}$, of χ_4 is an integer-valued character of degree 6. Show that χ_2 is a constituent of $\chi_{4,A}$, and that $\chi_3 := \chi_{4,A} - \chi_2$ is the fifth and final irreducible character of degree 3.

2. Consider the three integer-valued characters $\chi_{2,S}$, $\chi_{5,S}$, and $\chi_{4,A}$ of Question 1.

(i) It turns out that two of the three characters are the characters of permutation representations of A_5, and that the other one is not. Find which character is not the character of a permutation representation.

(ii) Of the two characters above that are characters of permutation representations, one has the property that its "fixed points minus one" character is irreducible, and the other does not. Find which is which.

3. Use Theorem 19.18 to prove that the dihedral group D_{16} of order 16 (described in Section 18.3) is not isomorphic to $D_8 \times C_2$. (Compare and contrast this result with Exercise 1.5 in the book.)