MATH 4140: Assignment 10

Chapters 17 and 18

1. Let G be the group given by the presentation

$$G = \langle c, d : c^4 = d^3 = 1,\ c^{-1}dc = d^{-1} \rangle.$$

You may assume without proof that G has order 12.

(i) Show that there is a representation $\rho : G \to GL(1, \mathbb{C})$ satisfying $\rho(c) = (i)$ and $\rho(d) = (1)$.

(ii) Use your answer to (i) to show that no two elements in the set $\{1, c, c^2, c^3, d, c^2d\}$ are conjugate, except possibly $\{1, d\}$ and $\{c^2, c^2d\}$.

(iii) Show that there is a surjective homomorphism $\sigma : G \to S_3$ satisfying $\sigma(c) = (12)$ and $\sigma(d) = (123)$.

(iv) Recall that S_3 has an irreducible character $\tilde{\chi}$ such that

$$\tilde{\chi}(1) = 2, \quad \tilde{\chi}((12)) = 0, \quad \text{and} \quad \tilde{\chi}((123)) = -1.$$

Show that this character lifts to a degree 2 irreducible character χ of G with the properties that $\chi(c) = 0$ and $\chi(d) = -1$. (This does not require any detailed calculations, just an explanation.)

(v) Use the character χ to show that 1 is not conjugate to d, and that c^2 is not conjugate to c^2d.

(vi) It turns out that $\{1, c, c^2, c^3, d, c^2d\}$ is a complete set of conjugacy class representatives of G. Prove that two rows of the character table of G are given as follows:

	1	c	c^2	c^3	d	c^2d
λ	1	i	-1	$-i$	1	-1
χ	2	0	2	0	-1	-1

(vii) By applying Proposition 17.14 repeatedly, using λ, construct the entire character table of G. (The book constructs the character table of G using different generators and a rather complicated argument. The complete table appears at the end of Chapter 18.)

[continued overleaf]

2. Let G be the group of order 12 whose character table was constructed in Question 1. Use the completed character table to answer the following questions.

(i) Show that the derived subgroup (commutator subgroup), G', of G has order 3. Show that the conjugacy class containing d is $\{d, d^2\}$, and that $G' = \{1, d, d^2\}$.

(ii) Show that the center, $Z(G)$, of G is given by $\{1, c^2\}$. Explain how one can tell, using the *rows* of the character table, that $Z(G)$ is cyclic.

(iii) Show that G has a normal subgroup N of order 6.

(iv) Use Proposition 17.5 to show that G has no proper nontrivial normal subgroups other than G', $Z(G)$, and N.