MATH 4140: Assignment 9

Chapter 15

1. Prove that the entries of the character table of the symmetric group S_n are all real. (It turns out that more is true: in fact, the entries of the character table are all integers.)

Chapter 16

2. Some of the irreducible characters of the symmetric group S_4 are given in the following table.

	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	-1	1

Here, χ_1 is the trivial character, and χ_3 and χ_4 are the characters described in Assignment 8, Question 6.

(i) Every symmetric group S_n ($n > 1$) has a character analogous to χ_2. Explain.

(ii) Show that the remaining character, χ_5, of S_4 has degree 2. Use the orthogonality relations to find the values of χ_5 on the conjugacy classes, and prove that χ_5 is not faithful.

3. Some of the irreducible characters of the alternating group A_5 are given in the following table.

	1	(123)	(12)(34)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	3	0	-1	α	β
χ_3	3	0	-1	β	α
χ_4	4	1	0	-1	-1

where $\alpha = (\sqrt{5}+1)/2$ and $\beta = (-\sqrt{5}+1)/2$. Here, χ_1 is the trivial character, and χ_2 and χ_3 are the characters appearing in Assignment 8, Question 7.

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(i) Explain where the character χ_4 comes from. (You do not need to show that χ_4 is irreducible.)

(ii) Show that the remaining irreducible character, χ_5, has degree 5. Use the orthogonality relations to find the values of χ_5 on the conjugacy classes.

(iii) There is a permutation representation $\rho : A_5 \to GL(6, \mathbb{C})$ with character χ in which (123) acts with no fixed points; in other words, we have $\chi((123)) = 0$. One of the entries of the character table gives a strong hint that this happens; explain which one.

(iv) Use the completed character table to show that every 5-cycle in A_5 is conjugate (in A_5) to its inverse.

4. The quaternion group Q_8 is a group of order 8 given by the presentation

$$Q_8 = \langle a, b : a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle.$$

The group Q_8 has five conjugacy classes, as follows:

$$\{1\}, \quad \{a^2\}, \quad \{a, a^{-1}\}, \quad \{b, b^{-1}\}, \quad \{ab, (ab)^{-1}\}.$$

(i) Observe that every element of Q_8 is conjugate to its inverse (because $a^{-2} = a^2$). Use this fact to prove that there can be no representation $\rho : Q_8 \to GL(1, \mathbb{C})$ with the property that $\rho(a) = (i)$.

(ii) Prove that $b^4 = 1$, and show that there are precisely four homomorphisms $\rho : Q_8 \to GL(1, \mathbb{C})$. Find the values of the corresponding four linear characters $\chi_1, \chi_2, \chi_3, \chi_4$ on the conjugacy classes.

(iii) Prove that the remaining irreducible character χ_5 of Q_8 has degree 2. Use the orthogonality relations to find the values of χ_5 on the conjugacy classes.
An important observation in character theory is that the groups D_8 and Q_8 have **the same character table despite being nonisomorphic groups**. This shows that there are limits to the information that can be deduced about a group from knowing its character table; in particular, one cannot always determine the isomorphism type of the group. It is also not possible to deduce how many elements there are of each order, because D_8 has two elements of order 4 and five elements of order 2, whereas Q_8 has six elements of order 4 and one element of order 2.

5. Let χ be a character of a group G with the property that $\chi(g)$ is a nonnegative real number for all $g \in G$.

(i) Use the row orthogonality relations to show that if χ is irreducible, then χ must be the trivial character.

(ii) Find an example of a nontrivial character χ with this property.