MATH 4140: Assignment 8

Chapter 13

1. Let $V = \mathbb{R}^3$ and let $T : V \to V$ be a rotation through an angle of θ that fixes an axis through the origin.

(i) Show that it is possible to find a basis of V with respect to which the matrix of T is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(ii) Show that the trace of T is equal to $2\cos\theta + 1$.

2. Recall that the alternating group A_4 has four conjugacy classes, represented by the elements 1, (12)(34), (123), and (132). The group of rotations of a regular tetrahedron (centered at the origin) is isomorphic to A_4. More specifically, the rotations are:

- the identity (1 element);
- rotations by $\pm 2\pi/3$ fixing an axis through a vertex and the midpoint of the opposite face (8 elements, corresponding to the 3-cycles);
- rotations by π fixing an axis through the midpoints of two opposite (skew) edges (3 elements, corresponding to the double transpositions).

Let $\rho_1 : A_4 \to GL(3, \mathbb{C})$ be the corresponding complex representation, in which each element $g \in A_4$ is represented by the (real-valued) matrix $\rho_1(g)$ of the corresponding rotation of the tetrahedron, and let χ_1 be the character of ρ_1.

Using the result of Question 1 (ii), show that the values of χ_1 on the four conjugacy classes of A_4 are $3, 0, 0, -1$, in some order, and match these four values to the conjugacy classes. [**Note:** This question and the next two do not require you to work out any of the matrices $\rho_1(g)$!]

3. Recall that the symmetric group S_4 has five conjugacy classes, represented by the elements 1, (12), (12)(34), (123), and (1234). The group of rotations of a cube (centered at the origin) is isomorphic to S_4. More specifically, the rotations are:

- the identity (1 element);
- rotations by $\pm 2\pi/3$ fixing an axis through two opposite vertices (8 elements, corresponding to the 3-cycles);
- rotations by π fixing an axis through the midpoints of two opposite (parallel) edges (6 elements, corresponding to the single transpositions);
- rotations by π fixing an axis through the midpoints of two opposite faces (3 elements, corresponding to the double transpositions);
- rotations by $\pi/2$ fixing an axis through the midpoints of two opposite faces (6 elements, corresponding to the 4-cycles).

[continued overleaf]

Let $\rho_2 : S_4 \to GL(3, \mathbb{C})$ be the corresponding complex representation, in which each element $g \in S_4$ is represented by the (real-valued) matrix $\rho_2(g)$ of the corresponding rotation of the cube, and let χ_2 be the character of ρ_2.
Using the result of Question 1 (ii), show that the values of χ_2 on the five conjugacy classes of S_4 are $3, 1, 0, -1, -1$, in some order, and match these five values to the conjugacy classes.

4. Recall that the alternating group A_5 has five conjugacy classes, represented by the elements 1, (12)(34), (123), (12345), and (12354). A dodecahedron is a regular solid with 12 pentagonal faces, 30 edges and 20 vertices, and the group of rotations of a dodecahedron (centered at the origin) is isomorphic to A_5. (This question can be adapted to work with the group of rotations of an icosahedron or of a traditional soccer ball, both of which are also A_5.) More specifically, the rotations are:

- the identity (1 element);
- rotations by $\pm 2\pi/3$ fixing an axis through two opposite vertices (20 elements, corresponding to the 3-cycles);
- rotations by π fixing an axis through the midpoints of two opposite (parallel) edges (15 elements, corresponding to the double transpositions);
- rotations by $\pm 2\pi/5$ fixing an axis through the midpoints of two opposite faces (12 elements, corresponding to some of the 5-cycles);
- rotations by $\pm 4\pi/5$ fixing an axis through the midpoints of two opposite faces (12 elements, corresponding to some of the 5-cycles).

(i) Use Question 1 to explain why a rotation by $\pm 2\pi/5$ cannot be similar to a rotation by $\pm 4\pi/5$, and therefore that the two types of 5-cycles in the list above must correspond to different conjugacy classes in A_5. **Hint:** You may find the following trigonometric identities helpful:

$$\cos(2\pi/5) = (\sqrt{5} - 1)/4; \qquad \cos(4\pi/5) = (-\sqrt{5} - 1)/4.$$

(ii) Let $\rho_3 : A_5 \to GL(3, \mathbb{C})$ be the corresponding complex representation, in which each element $g \in A_5$ is represented by the (real-valued) matrix $\rho_3(g)$ of the corresponding rotation of the dodecahedron, and let χ_3 be the character of ρ_3.
Using the result of Question 1 (ii), show that the values of χ_3 on the five conjugacy classes of A_5 are

$$3,\ 0,\ -1,\ (\sqrt{5} + 1)/2,\ (-\sqrt{5} + 1)/2,$$

in some order, and show that there are at most two possible ways to match these five values to the conjugacy classes.

(iii) It turns out that each of the two solutions in (ii) gives rise to a representation of A_5. Show that these two representations are not equivalent.

[continued overleaf]

Chapter 14

5. Consider the representation ρ_1 of A_4 and its associated character χ_1 from Question 2.
(i) Prove that $\langle \chi_1, \chi_1 \rangle = 1$, and deduce that χ_1 is irreducible. (Remember that there are *two* conjugacy classes of 3-cycles.)
(ii) Prove that χ_1 is equal to the "fixed points minus 1" character of A_4.
(iii) Prove that χ_1 is faithful.

6. Consider the representation ρ_2 of S_4 and its associated character χ_2 from Question 3.
(i) Prove that χ_2 is irreducible.
(ii) Let χ' be the "fixed points minus 1" character of S_4 of degree 3. Prove that χ' is irreducible.
(iii) Prove that χ_2 and χ' are distinct faithful characters of S_4.

7. Consider the two degree 3 characters of A_5 in Question 4, and denote the characters by χ_3 and χ_3'.
(i) Without calculation, explain why any nontrivial irreducible representation of A_5 over $\mathbb{C}$ must be faithful.
(ii) Prove that both χ_3 and χ_3' are irreducible characters of A_5.