MATH 4140: Assignment 7

Chapter 11

1. The group of rotations of a regular tetrahedron is isomorphic to the alternating group A_4. Because of this, A_4 has a degree 3 representation, and this representation turns out to be irreducible over $\mathbb{C}$. Use this fact to show that A_4 has precisely four nonisomorphic irreducible $\mathbb{C}A_4$-modules: one of dimension 3, and three of dimension 1.

2. Let G be the dihedral group D_{10}, given by the usual presentation

$$G = \langle a, b : a^5 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$

(i) Show that, up to equivalence, G has precisely two degree 1 representations over $\mathbb{C}$: the trivial representation, and the representation $\rho : G \to GL(1, \mathbb{C})$ satisfying $\rho(a) = (1)$ and $\rho(b) = (-1)$.

(ii) Use your answer to (i) to show that G has precisely four nonisomorphic irreducible $\mathbb{C}G$-modules: two of dimension 1, and two of dimension 2.

3. Let G be the dihedral group D_{12}, given by the usual presentation

$$G = \langle a, b : a^6 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle,$$

and suppose that $z \in \mathbb{C}$ satisfies $z^6 = 1$. Recall from Assignment 6 that there is a representation $\rho_z : G \to GL(2, \mathbb{C})$ satisfying

$$\rho_z(a) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \quad \text{and} \quad \rho_z(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and that ρ_z is irreducible if $z \notin \{+1, -1\}$.

(i) Let $z_1 = e^{2\pi i/6} = \cos(\pi/3) + i\sin(\pi/3)$, and let $z_2 = e^{2\pi i/3} = \cos(2\pi/3) + i\sin(2\pi/3)$. Show that $\rho_{z_1}(a)$ and $\rho_{z_2}(a)$ have different traces, and deduce that G has two inequivalent irreducible representations of degree 2 over $\mathbb{C}$.

(ii) Use your answer to (i) to show that G has precisely six nonisomorphic irreducible $\mathbb{C}G$-modules: two of dimension 2, and four of dimension 1.

(iii) Give an explicit description of the four irreducible representations of G over $\mathbb{C}$, as in Question 2 (i).

[continued overleaf]

Chapter 12

4. Maintain the notation of Question 3. Show that the matrices $\rho_{z_1}(a)$ and $\rho_{z_1}(a^2)$ are not similar, and deduce that a and a^2 are in different conjugacy classes of D_{12}.

5. Use the description of the conjugacy classes of A_4 given in the book to show that A_4 has no subgroup of order 6. [**Hint:** would such a subgroup be normal?]

6. Let G be the alternating group A_4. Use the description of the conjugacy classes of A_4 given in the book to prove the following:

(i) if $x \in G$ is a double transposition, then

$$C_G(x) = \{1, (12)(34), (13)(24), (14)(23)\};$$

(ii) if $x \in G$ is a 3-cycle, then $C_G(x)$ is the subgroup $\langle x \rangle$ generated by x.