MATH 4140: Assignment 6

Chapter 9

1. Suppose that $n \geq 3$, and let G be the dihedral group D_{2n}, given by the usual presentation
$$G = \langle a, b : a^n = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$
Suppose that $z \in \mathbb{C}$ satisfies $z^n = 1$.

(i) Show that there is a representation $\rho_z : G \to GL(2, \mathbb{C})$ satisfying
$$\rho_z(a) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \quad \text{and} \quad \rho_z(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(ii) Show that an element $g \in G$ is of the form $g = a^k$ for some k if and only if $\rho_z(g)$ is a diagonal matrix.

(iii) Prove that if $z \neq z^{-1}$ (i.e., if $z \notin \{+1, -1\}$) then the only matrices in $M_2(\mathbb{C})$ that commute with both $\rho_z(a)$ and $\rho_z(b)$ are scalar multiples of the identity. Prove also that this is not the case if $z = \pm 1$.

(iv) Use Schur's Lemma and your answer to (iii) to prove that ρ_z is irreducible if and only if $z \neq \pm 1$.

(v) Show that if $z = e^{2\pi i/n}$ then ρ_z is a faithful irreducible representation of G. Deduce that the center, $Z(G)$, of G is cyclic. [**Hint:** use your answer to (i).]

(vi) Use Schur's Lemma and your answer to (v) to show that the center of $Z(G)$ is equal to $\{1, a^{n/2}\}$ if n is even, and is equal to $\{1\}$ if n is odd.

2. Recall that there are two abelian groups of order 12: the cyclic group
$$C_{12} = \langle a : a^{12} = 1 \rangle,$$
and the group $C_2 \times C_6$. All representations in this question are defined over $\mathbb{C}$.

(i) Prove that there is a faithful irreducible representation of C_{12}, and explain how to construct one explicitly.

(ii) Explain why the only faithful irreducible representations of C_{12} have degree 1.

(iii) Prove that there is no faithful irreducible representation of $C_2 \times C_6$.

[continued overleaf]

Chapter 10

3. Let G be the symmetric group S_3, given by the presentation

$$G = \langle a, b : a^3 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle,$$

and define $u, v \in \mathbb{C}G$ by

$$u = 1 + a + a^2 + b + ab + a^2b$$

and

$$v = 1 + a + a^2 - b - ab - a^2b.$$

(i) Show that each of u and v spans a 1-dimensional $\mathbb{C}G$-submodule of $\mathbb{C}G$.
(ii) Show that the two submodules in (i) afford inequivalent representations.
(iii) Using the presentation, or otherwise, show that there are only two homomorphisms $\rho : G \to GL(1, \mathbb{C})$, and deduce that the two representations in (ii) are the only degree 1 representations of $G = S_3$, up to equivalence of representations.