MATH 4140: Assignment 5

Chapter 8

1. Let G be the cyclic group C_3 of order 3, given by the presentation

$$G = \langle x : x^3 = 1 \rangle.$$

You may assume without proof that there is a representation $\rho : G \to GL(2, \mathbb{C})$ satisfying

$$\rho(x) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$

Let $V = \mathbb{C}^2$ be the $\mathbb{C}G$-module that affords ρ, and let $\omega = e^{2\pi i/3}$ be a complex cube root of 1.

(i) Let V_1 be the subspace of V given by

$$V_1 = \left\{ \begin{bmatrix} a\omega \\ -a \end{bmatrix} : a \in \mathbb{C} \right\}.$$

Show that V_1 is a $\mathbb{C}G$-submodule of V, and find a complement to V_1 in V. ("A complement to V_1 in V" means "a submodule V_2 of V with the property that $V = V_1 \oplus V_2$". This means that you need to prove that your V_2 is a submodule.)

(ii) Now let $\rho' : G \to GL(2, \mathbb{R})$ be the representation of G over $\mathbb{R}$ given by the same matrix $\rho'(x) = \rho(x)$. Let $V' = \mathbb{R}^2$ be the corresponding $\mathbb{R}G$-module. Express V' as a direct sum of irreducible modules.

2. Let G be the dihedral group D_8, given by the usual presentation

$$G = \langle a, b : a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$

You may assume without proof that there is a representation $\rho : G \to GL(2, \mathbb{C})$ satisfying

$$\rho(a) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $V = \mathbb{C}^2$ be the $\mathbb{C}G$-module that affords ρ. Show that we have an isomorphism of $\mathbb{C}G$-modules

$$V \cong V_1 \oplus V_2,$$

where V_1 and V_2 are irreducible 1-dimensional $\mathbb{C}G$-modules, and find bases for each of V_1 and V_2.

[continued overleaf]

3. Let $G \leq GL_2(\mathbb{C})$ be the infinite group of matrices given by

$$G = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

Let $\rho : G \to GL_2(\mathbb{C})$ be the identity homomorphism, and let $V = \mathbb{C}^2$ be the $\mathbb{C}G$-module affording ρ.

(i) Show that G is isomorphic (as a group) to the group of integers, $\mathbb{Z}$.

(ii) Show that V has only one proper nontrivial submodule, U.

(iii) Show that U has no complement in V. (This shows that Maschke's Theorem is not true in general for infinite groups, even over $\mathbb{C}$.)

4. Let G be a finite group and let V and W be finite dimensional $\mathbb{C}G$-modules. Suppose that $\phi : V \to W$ is a surjective homomorphism of $\mathbb{C}G$-modules.

(i) Use Maschke's Theorem (and any other standard results that you need) to show that there is a submodule U of V with the property that V is isomorphic to $U \oplus \ker \phi$ as a $\mathbb{C}G$-module.

(ii) Let ϕ_U be the restriction of the function ϕ to the submodule U. (You may assume without proof that ϕ_U is a homomorphism of $\mathbb{C}G$-modules.) Show that $\phi_U : U \to W$ is an isomorphism of $\mathbb{C}G$-modules.

5. Let G be a finite group and let V be a finite dimensional $\mathbb{C}G$-module. Suppose that we have an isomorphism of $\mathbb{C}G$-modules

$$V \cong U_1 \oplus U_2 \oplus \cdots \oplus U_k,$$

where each of the U_i is 1-dimensional.

(i) Prove that it is possible to choose a basis of V so that V affords a representation ρ with the property that for all $g \in G$, $\rho(g)$ is a diagonal matrix.

(ii) Prove that if V is faithful then G is abelian.