MATH 4140: Assignment 4

Chapter 6

1. Let F be a field, let G be a finite group, and let $g \in G$ be an element of order $n > 1$. Show that, in the group algebra FG, we have

$$(1-g)(1+g+g^2+\cdots+g^{n-1}) = 0,$$

but that neither of the factors $(1-g)$ or $(1+g+g^2+\cdots+g^{n-1})$ is zero.

2. Let G be the dihedral group D_8, given by the usual presentation

$$G = \langle a, b : a^4 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle.$$

Show that the following identity holds in the group algebra $\mathbb{C}G$:

$$(ab + a^3b)(1+a) = (1+a)(b+a^2b).$$

Chapter 7

3. Recall from Assignment 2 that there is a presentation for the symmetric group S_3 given by

$$S_3 = \langle a, b : a^2 = b^2 = 1,\ (ab)^3 = 1\rangle.$$

You may assume without proof that there is a representation $\rho : S_3 \to GL(2, \mathbb{C})$ satisfying

$$\rho(a) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

[Note that we have $\rho(a) = -\theta(a)$ and $\rho(b) = -\theta(b)$, where θ is the homomorphism of Assignment 2, Question 4.] There is also a representation $\sigma : S_3 \to GL(3, \mathbb{C})$, which satisfies

$$\sigma(a) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma(b) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

[Note that $\sigma(a)$ and $\sigma(b)$ are the negatives of permutation matrices.] Let $V_\rho = \mathbb{C}^2$ and $V_\sigma = \mathbb{C}^3$ be the $\mathbb{C}S_3$-modules affording the representations ρ and σ respectively.

(i) Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

and let $\phi : \mathbb{C}^3 \to \mathbb{C}^2$ be the $\mathbb{C}$-linear map given by $\phi(v) = Av$. Show that ϕ is a homomorphism of $\mathbb{C}S_3$-modules.

[continued overleaf]

(ii) Prove that ϕ is surjective, and find kernel of ϕ as a submodule of $\mathbb{C}^3$.

(iii) Prove that $\ker(\phi)$ is not isomorphic to the trivial $\mathbb{C}S_3$-module.

(iv) Prove that ρ is equivalent to the representation θ of Assignment 2, Question 4.
Hint: try conjugating by the matrix

$$\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

(v) Prove that the $\mathbb{C}S_3$-module V_ρ is isomorphic to the module that affords the representation θ.