MATH 4140: Assignment 3

Chapters 4 and 5

1. Let G be the cyclic group of order 4, given by the presentation

$$G = \langle a : a^4 = 1 \rangle.$$

Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a 2-dimensional vector space over F with basis $\mathcal{B} = \{v_1, v_2\}$. It turns out that V can be made into a left FG-module in which the generator $a \in G$ acts as

$$a.v_1 = v_2 \quad \text{and} \quad a.v_2 = -v_1.$$

Let $\rho : FG \to GL_2(F)$ be the representation of G corresponding to this module.

(i) Calculate the matrix $\rho(a)$ (also known as $[a]_{\mathcal{B}}$).

(ii) Use your answer to (i) to calculate the matrices $\rho(a^2)$, $\rho(a^3)$ and $\rho(a^4) = \rho(1)$.

Hint: you should find that

$$\rho(a^3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(iii) Use your answer to (ii) to express the six module elements

$$a^2.v_1, \quad a^2.v_2, \quad a^3.v_1, \quad a^3.v_2, \quad 1.v_1, \quad 1.v_2$$

as linear combinations of v_1 and v_2.

2. Maintain the notation of Question 1 and set $F = \mathbb{R}$. Show that V is an irreducible FG-module. (Hint: consider the eigenvectors of $[a]_{\mathcal{B}}$.)

3. Maintain the notation of Question 1 and set $F = \mathbb{C}$. It turns out that the FG-module V has precisely two FG-submodules other than V itself and $\{0\}$.

(i) Find both these submodules.

(ii) Find a $\mathbb{C}$-basis $\mathcal{C}$ of V for which the matrices $[1]_{\mathcal{C}}$, $[a]_{\mathcal{C}}$, $[a^2]_{\mathcal{C}}$, and $[a^3]_{\mathcal{C}}$ are all diagonal matrices.