MATH 4140: Assignment 2

Chapter 3

1. Consider the dihedral group D_8 of order 8, given by the presentation

$$D_8 = \langle a, b : a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$

(i) Show that there is a representation $\rho : D_8 \to GL(2, \mathbb{C})$ satisfying

$$\rho(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Determine whether or not ρ is faithful.

2. Let G be the group given by the presentation

$$G = \langle a, b, c : a^2 = b^2 = c^2 = 1,\ ac = ca,\ (ab)^3 = 1,\ (bc)^3 = 1 \rangle.$$

Show that there is homomorphism of groups $\phi : G \to S_4$ satisfying

$$\phi(a) = (12), \quad \phi(b) = (23), \quad \text{and} \quad \phi(c) = (34).$$

3. It turns out that the homomorphism ϕ of Question 2 is an isomorphism, meaning that the presentation given in Question 2 is in fact a presentation for the symmetric group S_4. Assume that this is indeed the case, and define $a := (12)$, $b := (23)$ and $c := (34)$.

(i) Show that there is a representation $\theta : S_4 \to GL(2, \mathbb{C})$ satisfying

$$\theta(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \theta(b) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad \theta(c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) Show that θ is not faithful. (This does not require detailed calculations.)

4. There is a presentation for the symmetric group S_3 given by

$$S_3 = \langle a, b : a^2 = b^2 = 1,\ (ab)^3 = 1 \rangle.$$

(i) Show that there is a representation $\theta : S_3 \to GL(2, \mathbb{C})$, where $\theta(a)$ and $\theta(b)$ are as defined in Question 3(i).

(ii) Show that θ is faithful as a representation of S_3.

(iii) We also know that there is another presentation of S_3 given by

$$S_3 = \langle c, d : c^3 = d^2 = 1,\ d^{-1}cd = c^{-1} \rangle.$$

Without further calculation, explain why.