1. (40) Let $S = \mathbb{Q}^{<0}$ be the set of all strictly negative rational numbers. Define an operation $*$

$$a * b := -(a \times b),$$

where $\times$ is the usual way to multiply numbers.

(i) Show that $*$ is a binary operation on $S$, and that $(S, *)$ is an abelian group.

(ii) Let

$$H = \{-(2^k) : k \in \mathbb{Z}\} = \{\dots,\ -8,\ -4,\ -2,\ -1,\ -1/2,\ -1/4,\ -1/8,\ \dots\}$$

be the set of negatives of all integer powers of 2. Prove that $H$ is a subgroup of $S$.

2. (15) Let $(H, *)$ be the group in Question 1 (ii), and let $\mathbb{Z}$ be the group of integers under addition.

(i) Show that the function $\phi : \mathbb{Z} \to H$ defined by $\phi(r) = -(2^r)$ is an isomorphism. (You may assume without proof that $\mathbb{Z}$ is a group, and you do not need to define the term "isomorphism".)

(ii) Is the group $(H, *)$ cyclic? Why or why not?

3. (15) Let $GL(2, \mathbb{R})$ be the group of $2 \times 2$ invertible matrices with real entries under the operation of matrix multiplication, and let

$$A = \begin{pmatrix} 2 & -7 \\ 1 & -3 \end{pmatrix}.$$

(i) List the elements in the cyclic subgroup $K$ of $GL(2, \mathbb{R})$ that is generated by $A$.

(ii) Is $K$ isomorphic to the additive group $\mathbb{Z}_n$ of integers modulo $n$ for some $n > 1$? If so, find $n$. If not, explain why not.

4. (10) Let $(\mathbb{Z}_{20}, +_{20})$ be the group of integers modulo 20 under the operation of addition modulo 20. List all the subgroups of $\mathbb{Z}_{20}$, and draw the subgroup lattice.

5. (20) True or False. Mark with a "T" or an "F," and provide a brief explanation (a couple of lines), for each part.

(i) ______________Every group of order 4 is cyclic.

(ii) ______________Every subgroup of the group $\mathbb{Z}$ of integers is cyclic.

(iii) ______________The group $\mathbb{Z}$ has a subgroup with exactly two elements.

(iv) ________________ The dihedral group $D_3$ of order 6 is isomorphic to the symmetric group $S_3$.

(v) ________________ The dihedral group $D_{30}$ of order 60 is isomorphic to the symmetric group $S_n$ for some integer $n$.

(vi) ________________ There exists a nonabelian group of order 100.

**Name:** ______________________________

University of Colorado

Mathematics 3140: First In-Class Exam

September 25, 2024

| Problem | Points | Score |
|---|---|---|
| 1 | 40 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 10 | |
| 5 | 20 | |
| Total | 100 | |