1. (20) Let $V$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Recall that $V$ is a vector space with scalars $\mathbb{R}$, where vector addition and scalar multiplication are the usual (pointwise) addition and scalar multiplication of functions. Recall also that the zero vector of $V$ is the function $z(x) = 0$. Let

$$W = \left\{ f(x) \in V : \lim_{x \to -\infty} f(x) = 0 \right\};$$

in other words, $W$ is the set of functions that converge to zero as $x$ tends to $-\infty$. Prove that $W$ is a subspace of $V$. (You may use any standard properties of limits, provided that it is clear how you are using them.)

2. (20) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2 with real coefficients, together with the zero polynomial; in other words, a typical element of $\mathbb{P}_2$ is given by $a_0 + a_1 t + a_2 t^2$, where $a_0, a_1, a_2 \in \mathbb{R}$. Let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis for $\mathbb{P}_2$, and let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$. Let $T : \mathbb{P}_2 \to \mathbb{R}^2$ be the function defined by

$$T(\mathbf{p}(t)) = \begin{bmatrix} \mathbf{p}(1) \\ \mathbf{p}(2) \end{bmatrix},$$

so that for example we have $T(1 + 2t + 3t^2) = \begin{bmatrix} 6 \\ 17 \end{bmatrix}$. You may assume without proof that $T$ is a linear transformation.

(i) Find the matrix, $M$, of $T$ relative to the bases $\mathcal{B}$ and $\mathcal{E}$.

(ii) Find the rank of $M$, and show that the null space, $\mathrm{Nul}(M)$, has dimension 1.

(iii) Use your answer to (ii) to explain why $T$ is surjective ("onto" in the book).

(iv) Find a nonzero element of $\mathrm{Nul}(M)$.

(v) Using your answer to (iv), or otherwise, find a nonzero polynomial in $\ker(T)$.

3. (20) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose matrix with respect to the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is given by

$$B = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}.$$

(i) Prove that the matrix $B$ is orthogonal.

(ii) What does the fact that $B$ is orthogonal tell us about the transformation $T$?

(iii) It turns out that $T$ is a rotation in $\mathbb{R}^3$ by an angle $\alpha$ about a line $L$ through the origin. Using eigenvectors, or otherwise, find a vector in the direction of the axis of rotation $L$.

(iv) Write down the inverse of $B$.

(v) Use your answer to (iv) to find the angle $\alpha$.

4. (20) Let $C$ be the $4 \times 4$ matrix given by

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}.$$

(i) Show that $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is an eigenvector of $C$, and find the corresponding eigenvalue.

(ii) Write down another eigenvalue, $c$, of $C$, and find the dimension of the $c$-eigenspace.

(iii) Prove that $C$ is diagonalizable, and find a diagonal matrix $D$ that is similar to $C$. (You are not required to find an invertible matrix $P$ such that $C = PDP^{-1}$.)

(iv) Find the characteristic polynomial of $D$.

(v) Find the characteristic polynomial of $C$.

(vi) [For **20 bonus points** up to a maximum of 100.] Prove that there exists an orthogonal matrix $P$ such that $C = PDP^T$. (This does not require any complicated calculations. You are not required to find such a matrix, and there are no points for doing so.)

5. (20) Let

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ -8 \\ 5 \end{bmatrix}.$$

(i) Use the Gram–Schmidt process to find an orthogonal basis $\mathcal{B}$ for $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in such a way as to ensure that $\mathbf{x}_1 \in \mathcal{B}$ is one of the basis vectors. [You must use the Gram–Schmidt process to get full credit.]

(ii) Use your answer to (i) to find the dimension of $\mathrm{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, explaining your answer.

**Name:** ________________________________

University of Colorado

Mathematics 2135: Final Exam

August 9, 2024

| Problem | Points | Score |
| --- | --- | --- |
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |