1. (10) Let

$$M = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}, \quad \text{and} \;\; \mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}.$$

Use the fact that $M\mathbf{p} = \mathbf{b}$ to express the vector $\mathbf{b}$ as a linear combination of the columns of M.

2. (25) Let A be the matrix

$$A = \begin{bmatrix} 4 & 8 & 15 & 16 & 23 & 42 \\ 1 & 2 & 1 & 4 & 3 & 5 \\ 3 & 6 & 2 & 12 & 8 & 13 \end{bmatrix}.$$

Reduce the matrix A to reduced row echelon form. (Points will be deducted for not using elementary row operations.)

3. (25) Let A be the 3×6 matrix of Question 2. Use your answer to that problem to answer the following questions.

(i) Solve the equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

(ii) Is it possible to solve the equation $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^3$? (You do **not** need to find any examples of such vectors $\mathbf{b}$.)

4. (20) Determine whether the following sets of vectors are linearly independent or linearly dependent, giving a short explanation in each case. (Hint: you may be able to do this problem without doing any calculations.)

(i) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\};$

(ii) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 2 \end{bmatrix} \right\};$

(iii) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\};$

(iv) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix} \right\}.$

5. (20)

(i) Show that the map $S : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$S(x_1, x_2, x_3, x_4) = (x_1 + x_2, 0, x_2 + x_3, 2 + x_4)$$

is not linear.

(ii) Show that the map $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$T(x_1, x_2) = (4x_2 - 3x_1, 5x_1 - x_2, 0, 2x_2)$$

is linear, by finding a matrix that implements the mapping. (It is not enough simply to find the matrix.)

6. [For **20 bonus points** up to a maximum of 100.] Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set of vectors that span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a surjective linear transformation. (In the terminology of the book, this means that T is "onto".) Show that the vectors $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\}$ span $\mathbb{R}^m$.

Name: ______________________

University of Colorado

Mathematics 2135: First In-Class Exam

July 17, 2024

Problem	Points	Score
1	10	
2	25	
3	25	
4	20	
5	20	
Total	100	