1. (20) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$.

(i) Explain **why** it is obvious that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(ii) Write down the change of basis matrix, $P_{\mathcal{B}}$, from $\mathcal{B}$ to the standard basis $\mathcal{E}$ of $\mathbb{R}^n$, and calculate the inverse of $P_{\mathcal{B}}$.

(iii) Use your answer to (ii) to express the vector $\mathbf{x} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ as a linear combination of $\mathbf{b}_1$ and $\mathbf{b}_2$. (You need to show how the change of basis matrix is involved in order to get full credit.)

2. (20) Let V be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Recall that V is a vector space with scalars $\mathbb{R}$, where vector addition and scalar multiplication are the usual (pointwise) addition and scalar multiplication of functions. Recall also that the zero vector of V is the function $z(x) = 0$.

Let $W = \{f(x) \in V : f(-c) = f(c) \text{ for all } c \in \mathbb{R}\}$; in other words, W is the set of even functions in V. (For example, $\cos x \in W$, but $\sin x \notin W$.) Prove that W is a subspace of V. (There is no calculus in this question. If you use any properties of even functions beyond the definition, you need to justify them.)

3. (20) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2 with real coefficients, together with the zero polynomial; in other words, a typical element of $\mathbb{P}_2$ is given by $a_0 + a_1 t + a_2 t^2$, where $a_0, a_1, a_2 \in \mathbb{R}$. Let $T : \mathbb{P}_2 \to \mathbb{R}$ be the function defined by

$$T(\mathbf{p}(t)) = \mathbf{p}(2) - \mathbf{p}(1).$$

(i) Prove (carefully) that T is a linear transformation.

(ii) Find a **nonzero** element of the kernel, $\ker(T)$, of T.

(iii) Find a specific element $\mathbf{p}(t) \in \mathbb{P}_2$ for which $T(\mathbf{p}(t)) = \pi$. Describe the image of T, as an explicit subset of $\mathbb{R}$. ("Image" and "range" mean the same.)

4. (20) Use any method you like to find the determinant, $|A|$, of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and prove that A is invertible. (You are **not** required to find the inverse of A.)

5. (20) Let

$$B = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5\ \mathbf{v}_6] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

(i) The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ consisting of the first four columns of B is a linearly independent set. Why is this obvious from the answer to Problem 4?

(ii) Without further calculation, explain why the column space, $\text{Col}(B)$, of B is the whole of $\mathbb{R}^4$.

(iii) Explain why the dimension of the null space, $\text{Nul}(B)$, of B must be 2.

(iv) Notice that we have the following dependence relations between the columns of B: $\mathbf{v}_1 + \mathbf{v}_6 = \mathbf{v}_2 + \mathbf{v}_5 = \mathbf{v}_3 + \mathbf{v}_4$. Using these dependence relations, or otherwise, find two linearly independent vectors in $\text{Nul}(B)$, and explain why these two vectors must in fact be a basis for $\text{Nul}(B)$.

6. [For **20 bonus points** up to a maximum of 100.] Let V be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ defined in Question 2. Show that the set

$$\{e^{-x},\ \sin x,\ \cos x,\ e^{x}\}$$

is a linearly independent set in V.

Name: ______________________________

University of Colorado

Mathematics 2135: Second In-Class Exam

July 29, 2024

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	