Exploring Homology through the Particle Model

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Contents

Notation

B^n	The n-ball $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$
$\mathring{B}^n$	The open n-ball $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| < 1\}$
S^n	The n-sphere $\{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}$
I	The unit interval $[0, 1]$
$\mathbb{Z}_n$	The set of integers modulo n
$\cong$	Isomorphic groups or homeomorphic spaces

1 Introduction

A fundamental problem in topology is to determine whether two topological spaces are homeomorphic. To show that two spaces are homeomorphic, we need only demonstrate a homeomorphism between them. To show that two spaces are not homeomorphic is generally much harder or infeasible, as it requires verifying that no function between the two spaces is a homeomorphism. Thus, we rely on *topological invariants*, which are properties preserved by homeomorphisms. If two spaces are not homeomorphic, then they will differ in a topological invariant [1].

In this paper, we are concerned with studying a particular set of topological invariants called *homology groups*, which in some sense measure the number of holes in a given topological space. At a high level, we associate a topological space with a sequence of homology groups, where the nth homology group H_n measures the number of $(n+1)$-dimensional holes in that space.

As homology groups are algebraic objects, we will begin our study with an overview of topics in algebra that we will need in order to have a rigorous understanding of the structure of a homology group. We will then explore two methods for computing homology: simplicial homology, which requires us to cover simplicial complexes and triangulations, and cellular homology. After developing our understanding of the theory of homology, we will introduce a class of topological spaces called *Eilenberg-MacLane spaces*, and present the authors research on these spaces.

We assume that the reader has a working familiarity with abstract algebra (including both group and ring theory) and topology as covered at an undergraduate level.

2 Algebra

As a homology group will be defined as a quotient group, we choose to include the general construction of a quotient group for the reader to reference. We also choose to state the First Isomorphism Theorem, as it is integral to interpreting the quotient groups that will arise in our study of homology. Toward the end of the section, we introduce the idea of a module and a free abelian group, which we do not anticipate the reader to be familiar with. For fundamental topics in abstract algebra omitted here, we refer the reader to [6].

To define a quotient group, we first need to define what a coset is.

Definition 2.1. Let H be a subgroup of a group G. The **left coset of H with representative** $g \in G$ is the set $gH = \{gh \,|\, h \in H\}$. Similarly, the **right coset** is $Hg = \{hg \,|\, h \in H\}$ [6].

If G is abelian then $gH = Hg$ for all g and H, and so we will denote the cosets of H as gH.

Definition 2.2. A subgroup H of a group G is a **normal subgroup** if $gH = Hg$ for all $g \in G$ [6].

Notice that we immediately get that every subgroup of an abelian group is normal. In our study of homology we will exclusively work with abelian groups.

Definition 2.3. Given a normal subgroup N of a group G, the **quotient group** of G and N is $G/N = \{gN \,|\, g \in G\}$ with the operation $(aN)(bN) = abN$ [6].

Example 2.4. Consider the normal subgroup $2\mathbb{Z}$ of the group $\mathbb{Z}$. Then

$$\mathbb{Z}/2\mathbb{Z} = \{0 + 2\mathbb{Z}, 1 + 2\mathbb{Z}\} \cong \mathbb{Z}_2.$$

We can see that these two groups are in fact isomorphic since $\phi : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}_2$ defined by $\phi(a + 2\mathbb{Z}) = a$ is clearly a group isomorphism.

As stated earlier, the following theorem will be integral to systematically interpreting the quotient structure of a homology group.

Theorem 2.5 (The First Isomorphism Theorem). *If $\psi : G \to H$ is a group homomorphism, then $\ker \psi$ is a normal subgroup of G. Then $G/\ker \psi \cong \operatorname{im} \psi$ [6].*

The First Isomorphism Theorem is also very powerful for representing strange or complicated quotient structures more succinctly. We illustrate this with an example.

Example 2.6. Consider the group $\mathbb{Z} \times \mathbb{Z}$ and subgroup $2\mathbb{Z} \times \{0\}$. Since $\mathbb{Z} \times \mathbb{Z}$ is abelian, $2\mathbb{Z} \times \{0\}$ is normal.

First we will interpret the quotient group of $\mathbb{Z} \times \mathbb{Z}$ and $2\mathbb{Z} \times \{0\}$ using only the definition. The set of cosets of $2\mathbb{Z} \times \{0\}$ in $\mathbb{Z} \times \mathbb{Z}$ is $\{(a, b) + 2\mathbb{Z} \times \{0\} \mid (a, b) \in \mathbb{Z} \times \mathbb{Z}\}$. Since we add elements of a direct product componentwise, we may also write this set as $\{\{(a + 2n, b) \mid n \in \mathbb{Z}\} \mid a, b \in \mathbb{Z}\}$. However, as we have established, $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$, and so we may write this set of cosets as $\{\{(0 + 2n, b) \mid n \in \mathbb{Z}\} \mid b \in \mathbb{Z}\} \cup \{\{(1 + 2n, b) \mid n \in \mathbb{Z}\} \mid b \in \mathbb{Z}\}$. We may again rewrite this as $\{(0 + 2\mathbb{Z}) \times \{b\} \mid b \in \mathbb{Z}\} \cup \{(1 + 2\mathbb{Z}) \times \{b\} \mid b \in \mathbb{Z}\}$. We claim that this is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$. Consider the homomorphism

$$\phi : \frac{\mathbb{Z} \times \mathbb{Z}}{2\mathbb{Z} \times \{0\}} \to \mathbb{Z}_2 \times \mathbb{Z}$$

defined by $\phi\left((a + 2\mathbb{Z}) \times \{b\}\right) = (a, b)$. It follows that ϕ is an isomorphism, since $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$ and the identity homomorphism is an isomorphism. Thus

$$\frac{\mathbb{Z} \times \mathbb{Z}}{2\mathbb{Z} \times \{0\}} \cong \mathbb{Z}_2 \times \mathbb{Z}.$$

Now we will use the First Isomorphism Theorem to reach the same conclusion without so much dirty work. We construct the homomorphism $\psi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}$ where

$$\psi(a, b) = \begin{cases} (0, b) & \text{if } a \text{ is even,} \\ (1, b) & \text{if } a \text{ is odd.} \end{cases}$$

Now, $\ker \psi = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid \psi(a, b) = (0, 0)\} = \{(2n, 0) \mid n \in \mathbb{Z}\} = 2\mathbb{Z} \times \{0\}$ and $\operatorname{im} \psi = \mathbb{Z}_2 \times \mathbb{Z}$. By the First Isomorphism Theorem, we may conclude that

$$\frac{\mathbb{Z} \times \mathbb{Z}}{2\mathbb{Z} \times \{0\}} \cong \mathbb{Z}_2 \times \mathbb{Z}.$$

The second method was far simpler and more elegant.

2.1 Modules and Free Abelian Groups

A vector space is defined as an additive abelian group with a scalar product that is distributive, associative, and respects the multiplicative identity, where the scalars come from a field [6]. If instead we allow our scalars to come from an arbitrary ring, our construction is called a module. In general, we may use a ring that is not commutative, in which case we differentiate between left and right modules, or a ring that does not have a multiplicative identity. However, we will only concern ourselves with commutative rings with unity. For a thorough discussion of module theory, we refer the reader to [5].

Definition 2.7. Let R be a commutative ring with unity and M be an additive abelian group. We say that M is an R**-module** if there is a notion of scalar multiplication such that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$ the following are satisfied:
(i) $r(m_1 + m_2) = rm_1 + rm_2$;
(ii) $(r_1 + r_2)m_1 = r_1 m + r_2 m_1$;
(iii) $(r_1 r_2)m_1 = r_1(r_2 m_1)$;
(iv) $1m_1 = m_1$, where 1 is the multiplicative identity in R [5].

As with vector spaces, we will refer to the elements of R as **scalars**. The reader may also find it useful to think of the elements of M as vectors. A **submodule** is defined as expected: a subset of a module that is a module in its own right.

Example 2.8. If G is an additive abelian group, then G admits a scalar multiplication such that if $n \in \mathbb{Z}$ and $g \in G$ then ng denotes $\sum_{i=1}^{n} g$. It is clear that the necessary conditions for G to be a $\mathbb{Z}$-module are satisfied. Thus every additive abelian group is a $\mathbb{Z}$-module.

Example 2.9. Let R be a commutative ring with unity. Then R^n is an R-module with vector addition defined as component-wise addition of elements in R^n and scalar multiplication defined as multiplying each component by the scalar.

Since modules are a generalization of vector spaces, we have a natural generalization of the idea of a basis to modules.

Definition 2.10. Let M be an R-module and let $S = \{m_1, m_2, \ldots, m_n\}$ be a subset of M. The **submodule of** M **generated by** S is the set of all linear combinations of vectors in S with coefficients from R, and will be denoted

$$\langle S \rangle = \langle m_1, m_2, \ldots, m_n \rangle := \{r_1 m_1 + r_2 m_2 + \cdots + r_n m_n \,|\, r_i \in R\}.$$

If $S = \emptyset$, then $\langle S \rangle := \{0\}$. We say that S is **linearly independent** if the equation $r_1 m_1 + r_2 m_2 + \cdots + r_n m_n = 0$ has only the trivial solution, $r_i = 0$ for all i. We say that S **spans** M if $\langle S \rangle = M$. We say that S is a **basis** for M if it is linearly independent and spans M [5].

Definition 2.11. We say that an R-module M is **free** if it has a basis [5].

Definition 2.12. Let G be an additive abelian group with a basis when viewed as a $\mathbb{Z}$-module. Then we call G a **free abelian group** [5].

We will see many examples of free abelian groups in the coming sections, so we omit examples of them for the time being and instead give an interesting nonexample.

Example 2.13. Not every additive abelian group is free. Let G be a finite abelian group of order n. Then for $g \in G$, $ng = 0$ and so $\{g\}$ is a linearly dependent set. Thus G has no basis.

The distinction between free module and free abelian group may seem unnecessary. Indeed, one may use the terms free abelian group and free $\mathbb{Z}$-module interchangeably. We have chosen to include this distinction as "free abelian group" is most commonly used in the literature on homology and, as such, we will use the terminology to match the literature. We have also made this choice to emphasize that free abelian groups behave like vector spaces over $\mathbb{Z}$, and so we may use the theory of linear algebra in computations in a free abelian group, taking care that we do not have multiplicative inverses for our scalars.

3 Homology Primer

Homology theory assigns to a topological space X a sequence of abelian groups

$$H_0(X), H_1(X), H_2(X), \ldots,$$

where the nth homology group in some sense measures the number of $(n+1)$-dimensional holes in X. In general, a 1-dimensional hole corresponds to two vertices that cannot be connected by a path in X. A cannonical example of a space with an $(n+1)$-dimensional hole is S^n for $n \geq 1$.

To build a bit of intuition for how we think of detecting $(n+1)$-dimensional holes in a space, let us consider how to detect 2-dimensional holes. We think of placing all possible loops on our space and check whether those loops bound some 2-dimensional component of our space. If a loop is not the boundary of some 2-dimensional component of our space, then it bounds a 2-dimensional hole in our space.

4 Triangulations

For homology theory to be of much use to us, we need to be able to actually compute the homology group for a given topological space. To compute the homology of a given space, our general strategy will be to work with a topologically equivalent space that has some additional structure that makes homology computations feasible. The first class of topological spaces we will work with are those that are triangulable, or admit a triangulation. To build up our theory of triangulations, we will need to first define what a simplex is.

Definition 4.1. Let $v_0, v_1, \ldots, v_k$ be points in $\mathbb{R}^n$. If $v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0$ are linearly independent, then we say that $v_0, v_1, \ldots, v_k$ are in **general position** [3].

Definition 4.2. Let $v_0, v_1, \ldots, v_k \in \mathbb{R}^n$ be in general position. The set

$$\left\{ x = \sum_{i=0}^{k} \lambda_i v_i \,|\, \lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

is called a k**-simplex**, and $v_0, v_1, \ldots, v_k$ the **vertices** of the simplex. For simplices A and B, if the set of vertices of B is a subset of the set of vertices of A, we say B is a **face** of A [3].

Example 4.3. Consider the points $(1, 0, 0), (0, 1, 0), (0, 0, 1) \in \mathbb{R}^3$. We have $(0, 1, 0) - (1, 0, 0) = (-1, 1, 0)$ and $(0, 0, 1) - (1, 0, 0) = (-1, 0, 1)$, which are clearly linearly independent. Thus our points are in general position and so we can build a 2-simplex with vertices $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. This simplex is shown in Figure 1.

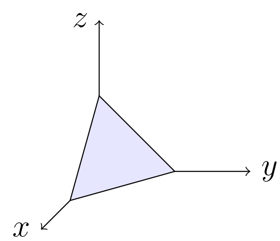


Figure 1

Example 4.4. In general, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a solid triangle, and a 3-simplex is a solid tetrahedron, as illustrated in Figure 2.

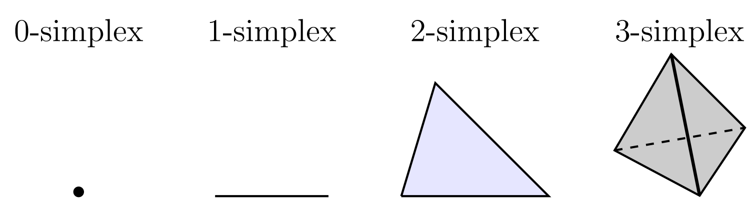


Figure 2

Definition 4.5. Let K be a finite collection of simplices in $\mathbb{R}^n$. We say that K is a **simplicial complex** if the following conditions are satisfied.

(i) For every simplex in K, each of its faces is also a simplex in K.
(ii) Whenever two simplices in K intersect, their intersection is a single face of each of them [9].

We may denote a collection of simplices with addition, so that if σ and τ are two simplices in a simplicial complex, then $\sigma + \tau$ denotes the simplicial complex consisting of σ and τ. This addition is commutative.

Example 4.6. In Figure 3, the left collection of simplices is a simplicial complex since the intersection of any two simplices is a face of both of them, while the right collection fails to be a simplicial complex as it does not have this property.

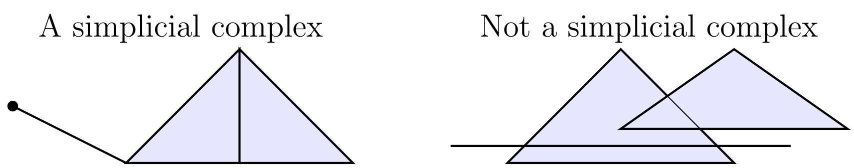


Figure 3

Example 4.7. The tetrahedron is a simplicial complex, regardless of whether we think of it as a surface or a solid. If we view the octahedron and icosahedron as surfaces, they are also simplicial complexes.

Example 4.8. Note that the simplices in a simplicial complex need not intersect at all. Figure 4 illustrates such a complex.

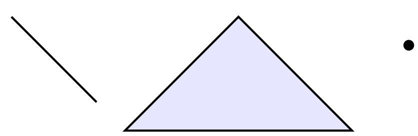


Figure 4

Definition 4.9. Let K be a simplicial complex in $\mathbb{R}^n$. If we give K the subspace topology from $\mathbb{R}^n$, we call the resulting topological space a **polyhedron** and denote it $|K|$ [3].

Definition 4.10. A **triangulation** of a topological space X is a simplicial complex K together with a homeomorphism $h : |K| \to X$. If there is a triangulation of X, then we say that X is **triangulable** [3].

Example 4.11. Since the tetrahedron, octahedron, and icosahedron as surfaces are homeomorphic to S^2, each gives a triangulation of the 2-sphere. The following figure shows a triangulation of S^2 induced by the octahedron.

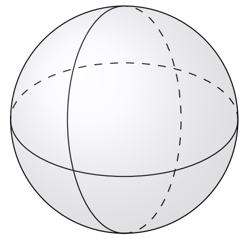


Figure 5: A triangulation of S^2.

Example 4.12. If we wish to triangulate a space that can be constructed as a quotient space of a regular n-gon, we will represent that triangulation with a triangulation of the n-gon. Figure 6 depicts a triangulation of the torus.

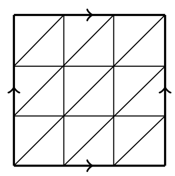


Figure 6: A triangulation of the torus.

5 Simplicial Homology

Now that we have an understanding of what a simplicial complex is, we can develop our homology theory for simplicial complexes.

Definition 5.1. Let $v_0, v_1, \ldots, v_k$ be the vertices of a k-simplex. We define an equivalence relation $\sim$ on the set of orderings of $v_0, v_1, \ldots, v_k$ such that two orderings are equivalent if they differ in an even permutation. We call an equivalence class of $\sim$ an **orientation** of the k-simplex. An **oriented simplex**, usually denoted σ or τ, is a simplex together with a choice of orientation. We specify the orientation of an oriented k-simplex σ by writing σ as a $(k+1)$-tuple that is a representative for its orientation class. We use the symbol $-\sigma$ to denote σ with opposite orientation [9].

It follows that every k-simplex, for $k > 0$, has exactly two orientations, and a 0-simplex has only one orientation since there is only one way to order one element.

Example 5.2. In Figure 7, we give cannonical examples for the orientations of a 0,1,2, and 3-simplex.

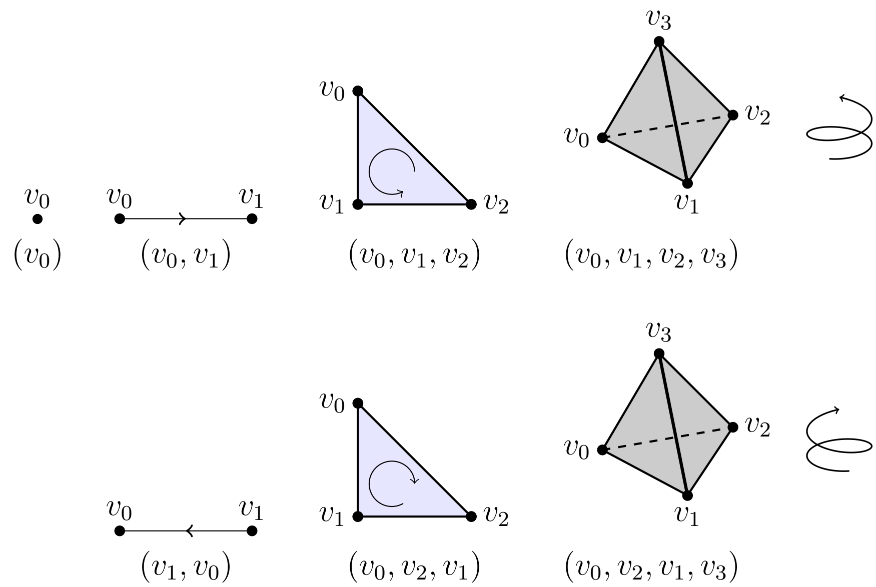


Figure 7

Definition 5.3. The **boundary** of an oriented k-simplex $\sigma = (v_0, v_1, \dots, v_k)$ is

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i (v_0, \dots, \hat{v}_i, \dots, v_k),$$

where $(v_0, \dots, \hat{v}_i, \dots, v_k)$ is the k-cycle obtained by deleting v_i from $(v_0, v_1, \dots, v_k)$. The boundary of a point is defined to be 0 [9].

Example 5.4. Let v_0, v_1, v_2, v_3 be points in $\mathbb{R}^4$ in general position. We will compute the boundaries of the oriented simplices (v_0, v_1), (v_0, v_1, v_2), and (v_0, v_1, v_2, v_3). We will use the fact that, for an oriented simplex σ, $-\sigma$ is σ with orientation reversed so that our boundaries will have only positive coefficients. We have

$$\begin{aligned}
\partial(v_0, v_1) &= v_1 - v_0, \\
\partial(v_0, v_1, v_2) &= (v_1, v_2) - (v_0, v_2) + (v_0, v_1) = (v_0, v_1) + (v_1, v_2) + (v_2, v_0), \\
\partial(v_0, v_1, v_2, v_3) &= (v_1, v_2, v_3) - (v_0, v_2, v_3) + (v_0, v_1, v_3) - (v_0, v_1, v_2) \\
&= (v_0, v_1, v_3) + (v_0, v_2, v_1) + (v_0, v_3, v_2) + (v_1, v_2, v_3).
\end{aligned}$$

Intuitively, we can think of the boundary of an oriented k-simplex as the sum of its $(k-1)$ dimensional faces, each with orientation induced from the original simplex. Figure 8 illustrates this, and gives a geometric interpretation for our above calculations.

Now that we have an understanding of oriented simplices and their boundaries, we can use

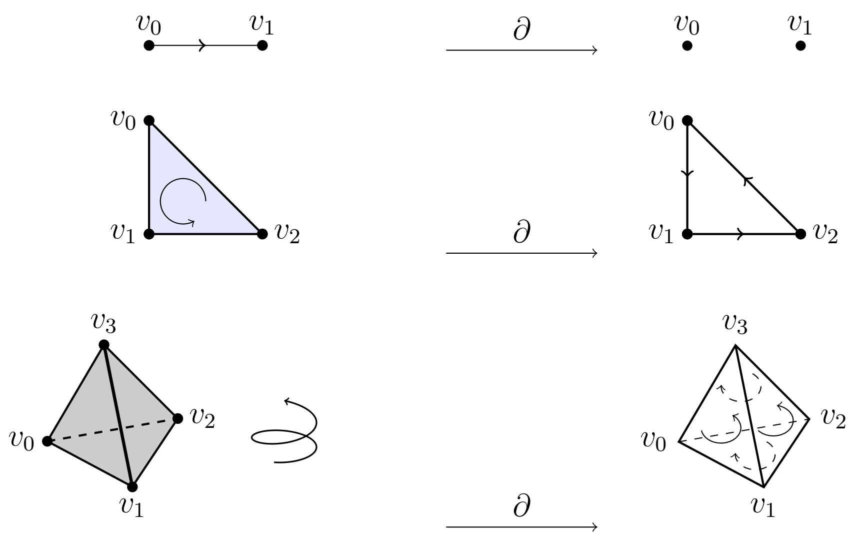


Figure 8

these notions to associate a given simplicial complex with a sequence of algebraic structures. Doing so loses some of the geometric interpretations that we have explored, but it also allows us to analyze a triangulable space algebraically.

Definition 5.5. Let K be a simplicial complex. We call the free abelian group generated by the oriented n-simplices of K, subject to the relation $\sigma + (-\sigma) = 0$, the n**th chain group** of K and denote it $C_n(K)$. We call an element of $C_n(K)$ an n**-chain**. If $\sigma \in C_n(K)$ can be written as a linear combination of oriented n-simplices of K with coefficients in $\{-1, 0, 1\}$, then we say that σ is an **elementary** n**-chain** [3].

Definition 5.6. We extend the definition of the boundary of an oriented n-simplex linearly to define the n**th boundary homomorphism** $\partial_n : C_n(K) \to C_{n-1}(K)$. We defined $C_{-1}(K) = \{0\}$. We call $\ker \partial_n$ the **group of** n**-cycles** of K. Unsurprisingly, an element of $\ker \partial_n$ is called an n**-cycle**. We call $\operatorname{im} \partial_{n+1}$ the **group of bounding** n**-cycles**. An element of $\operatorname{im} \partial_{n+1}$ is called a **bounding** n**-cycle** [3].

Note that in the literature it is common to denote $\ker \partial_n$ by $Z_n(K)$ and $\operatorname{im} \partial_{n+1}$ by $B_n(K)$.

Lemma 5.7. *For a simplicial complex K, $\partial_{n-1} \circ \partial_n$ is the zero homomorphism.*

Proof. Let $(v_0, v_1, \ldots, v_n)$ be an oriented n-simplex in K. From Definition 5.3, we have

$$\begin{aligned}
\partial_{n-1} \circ \partial_n(v_0, v_1, \ldots, v_n) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i(v_0, \ldots, \hat{v}_i, \ldots, v_n)\right) \\
&= \sum_{i=0}^{n}(-1)^i\partial_{n-1}(v_0, \ldots, \hat{v}_i, \ldots, v_n) \\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{i-1}(-1)^j(v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n) \\
&\quad + \sum_{i=0}^{n}(-1)^i\sum_{j=i+1}^{n}(-1)^{j-1}(v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i-1}\left((-1)^{i+j} + (-1)^{i+j-1}\right)(v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n) \\
&= 0.
\end{aligned}$$

Suppose that $C_n(K) = \langle \sigma_0, \sigma_1, \ldots, \sigma_k \rangle$. Then for any element of $C_n(K)$, we have

$$\partial_{n-1} \circ \partial_n\left(\sum_{i=0}^{k}\alpha_i\sigma_i\right) = \sum_{i=0}^{k}\alpha_i(\partial_{n-1} \circ \partial_n\sigma_i) = \sum_{i=0}^{k}\alpha_i \cdot 0 = 0,$$

where $\alpha_i \in \mathbb{Z}$ for each i. Thus $\partial_{n-1} \circ \partial_n$ is the 0 homomorphism. □

Definition 5.8. By the previous lemma, $\operatorname{im}\partial_{n+1}$ is a subgroup of $\ker\partial_n$, and so the quotient group $\ker\partial_n/\operatorname{im}\partial_{n+1}$ is defined. Thus, we define the **nth homology group of** K to be $H_n(K) = \ker\partial_n/\operatorname{im}\partial_{n+1}$ [3].

Definition 5.9. For $z_1, z_2 \in \ker\partial_n$, we say that z_1 is **homologous** to z_2 if $z_1 - z_2 = z$ for some $z \in \operatorname{im}\partial_{n+1}$ [3].

This gives us an equivalence relation on the n-cycles of a simplicial complex, and the set of representatives for these equivalence classes generate the nth homology group.

We now state theorems that we will find relevant to our study of homology theory.

Theorem 5.10. *If two triangulable spaces X and Y are homeomorphic, then they have the same sequence of homology groups.*

Proof. For a proof of this theorem we refer the reader to Chapter 2 in [9]. □

Thus we may compute the homology groups for a triangulable space X by computing the homology groups of a simplicial complex that is a triangulation of X.

Theorem 5.11. *If K is a simplicial complex with no n-simplices, then $H_n(K) = \{0\}$ and $H_{n-1}(K) = \ker \partial_{n-1}$.*

Proof. Since K has no n-simplices, we have $C_n(K) = \{0\}$, and so $\ker \partial_n = \operatorname{im} \partial_n = \{0\}$. By Lemma 5.7, we have $\operatorname{im} \partial_{n+1} = \{0\}$. Therefore

$$H_n(K) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{\{0\}}{\{0\}} = \{0\}\,,$$

and

$$H_{n-1}(K) = \frac{\ker \partial_{n-1}}{\operatorname{im} \partial_n} = \frac{\ker \partial_{n-1}}{\{0\}} = \ker \partial_{n-1}.$$

□

Theorem 5.12. *Let K be a simplicial complex. The rank of $H_0(K)$ is the number of path components in K.*

Proof. We will first prove that K is path connected if and only if $H_0(K) \cong \mathbb{Z}$.

($\Rightarrow$) Suppose that K is path connected. Fix a vertex v in K. Then for any other vertex w in K, there is a path from v to w in K. Since K is a simplicial complex, this path is necessarily a sequence of edges in K of the form $\sigma = (v, v_1) + (v_1, v_2) + \cdots + (v_k, w)$. Then $\partial_1 \sigma = w - v$. By Definition 5.9, v and w are homologous vertices. Since this holds for any vertex w of K, we have that all vertices in K are homologous and so there is only one homology class for the vertices of K. Therefore $H_0(K) = \langle v \rangle \cong \mathbb{Z}$.

($\Leftarrow$) Suppose that $H_0(K) \cong \mathbb{Z}$. Then there is only one homology class on the vertices of K by Definition 5.9. So, for any pair of vertices v, w of K, there is some $\sigma \in C_1(K)$ where $\partial_1 \sigma = w - v$. Without loss, we may assume that σ is an elementary 1-chain. Then σ is a path from v to w, and so K is path connected.

Suppose now that K has path components P_i, where $1 \leq i \leq \ell$. Since each path component is path connected, it follows that $H_0(P_i) \cong \mathbb{Z}$ for each i. It follows from Definition 5.9 that $H_0(P_i \cup P_j) \cong \mathbb{Z}^2$ for any i, j where $i \neq j$. Therefore, since $K = \bigcup_{i=1}^{\ell} P_i$, we have $H_0(K) \cong \mathbb{Z}^\ell$. □

5.1 Computing the Homology of Simplicial Complexes

Now that we have developed the theory of simplicial homology, we may demonstrate how to use simplicial homology to analyze topological spaces. We begin with a careful analysis of of S^1 to illustrated the mechanics of computing homology groups. We then analyze B^2 and use our computations to conclude that S^1 and B^2 are not homeomorphic. Our third example illustrates Theorem 5.12, as well as the fact that two spaces with the same homology groups need not be homeomorphic.

Example 5.13.

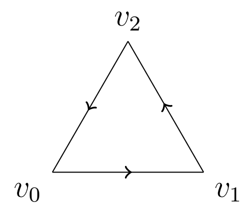


Figure 9: A triangulation of S^1.

Let K be the simplicial complex homeomorphic to S^1 given in Figure 9. The chain groups for K are $C_0(K) = \langle v_0, v_1, v_2\rangle$, $C_1(K) = \langle (v_0, v_1), (v_1, v_2), (v_2, v_0)\rangle$, and $C_n(K) = \{0\}$ for $n \geq 2$ since K has no simplices of dimension larger than 1. By definition, $\partial_0 v_0 = \partial_0 v_1 = \partial_0 v_2 = 0$, and so $\ker \partial_0 = \langle v_0, v_1, v_2\rangle$ and $\operatorname{im} \partial_0 = \{0\}$. Computing $\partial_1 C_1(K)$, we have $\partial_1(v_0, v_1) = v_1 - v_0$, $\partial_1(v_1, v_2) = v_2 - v_1$, and $\partial_1(v_2, v_0) = v_0 - v_2$. If we order the bases of $C_1(K)$ and $C_0(K)$ as they are listed above, we can represent ∂_1 with the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

which we can see has rank 2 and nullity 1, which tells us that $\ker \partial_1 \cong \mathbb{Z}$ and $\operatorname{im} \partial_1 \cong \mathbb{Z}^2$. We can see from the reduced row echelon form of A that $\ker \partial_1 = \langle (v_0, v_1) + (v_1, v_2) + (v_2, v_0)\rangle$ and $\operatorname{im} \partial_1 = \langle v_1 - v_0, v_2 - v_1\rangle$. Since $C_n(K) = \{0\}$ for $n \geq 2$, we have $\ker \partial_n = \operatorname{im} \partial_n = \{0\}$.

Now we may compute the homology groups of K. We have

$$H_0(K) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle v_0, v_1, v_2\rangle}{\langle v_1 - v_0, v_2 - v_1\rangle},$$

which we can evaluate by first changing basis, so that $\langle v_0, v_1, v_2\rangle = \langle v_0, v_1 - v_0, v_2 - v_1\rangle$. Now, consider the group homomorphism $\phi : \langle v_0, v_1 - v_0, v_2 - v_1\rangle \to \langle v_1 - v_0, v_2 - v_1\rangle$ defined by $\phi(a(v_0) + b(v_1 - v_0) + c(v_2 - v_1)) = a(v_0)$, having $\ker \phi = \langle v_1 - v_0, v_2 - v_1\rangle$ and $\operatorname{im} \phi = \langle v_0\rangle$. By the First Isomorphism Theorem we have $\langle v_0, v_1 - v_0, v_2 - v_1\rangle / \langle v_1 - v_0, v_2 - v_1\rangle \cong \langle v_0\rangle$. Thus

$$H_0(K) = \frac{\langle v_0, v_1 - v_0, v_2 - v_1\rangle}{\langle v_1 - v_0, v_2 - v_1\rangle} \cong \langle v_0\rangle \cong \mathbb{Z},$$

which tells us that K is connected. By a similar argument, we have

$$H_1(K) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle (v_0, v_1) + (v_1, v_2) + (v_2, v_0) \rangle}{\{0\}} \cong \langle (v_0, v_1) + (v_1, v_2) + (v_2, v_0) \rangle \cong \mathbb{Z},$$

which tells us that K has one 1-dimensional hole in it. Finally, $H_n(K) = \{0\}$ for $n \geq 2$, and so K has no holes of dimension greater than 1.

Example 5.14.

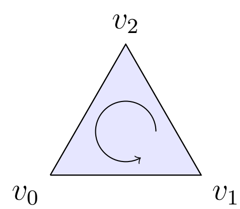


Figure 10: A triangulation of B^2.

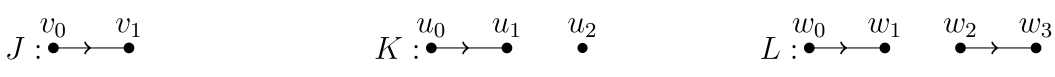
Let K be the simplicial complex homeomorphic to B^2 in Figure 10. Since the zeroth and first chain complexes are the same for this simplicial complex as the previous example, we have $\ker \partial_0 = \langle v_0, v_1, v_2 \rangle$, $\operatorname{im} \partial_0 = \{0\}$, $\ker \partial_1 = \langle (v_0, v_1) + (v_1, v_2) + (v_2, v_0) \rangle$, and $\operatorname{im} \partial_1 = \langle v_1 - v_0, v_2 - v_1 \rangle$. The only difference here is that $C_2(K) = \langle (v_0, v_1, v_2) \rangle$, which gives us that $\partial_2(v_0, v_1, v_2) = (v_0, v_1) + (v_1, v_2) + (v_2, v_0)$, and so $\ker \partial_2 = \{0\}$ and $\operatorname{im} \partial_2 = \langle (v_0, v_1) + (v_1, v_2) + (v_2, v_0) \rangle$. Thus, the homology groups for K are

$$\begin{aligned} H_0(K) &= \langle v_0 \rangle \cong \mathbb{Z} \\ H_1(K) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle (v_0, v_1) + (v_1, v_2) + (v_2, v_0) \rangle}{\langle (v_0, v_1) + (v_1, v_2) + (v_2, v_0) \rangle} \cong \{0\} \\ H_n(K) &= \{0\} \quad \text{for } n \geq 2. \end{aligned}$$

So we may conclude that B^2 is connected with no holes of dimension one or greater.

From the previous two examples we may conclude that $S^1 \ncong B^1$ since $H_1(S_1) \cong \mathbb{Z}$ while $H_1(B_2) = \{0\}$.

Example 5.15. Consider the following three spaces.



For J, we have $C_0(J) = \langle v_0, v_1 \rangle$ and $C_1(J) = \langle (v_0, v_1) \rangle$. Computing the boundary maps, we have $\partial_0 v_0 = \partial_0 v_1 = 0$ and $\partial_1(v_0, v_1) = v_1 - v_0$. Thus

$$H_0(J) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle v_0, v_1 \rangle}{\langle v_1 - v_0 \rangle} = \frac{\langle v_0, v_1 - v_0 \rangle}{\langle v_1 - v_0 \rangle} \cong \langle v_0 \rangle \cong \mathbb{Z}$$
$$H_1(J) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\{0\}}{\{0\}} = \{0\} .$$

For K, we have $C_0(K) = \langle u_0, u_1, u_2 \rangle$ and $C_1(K) = \langle (u_0, u_1) \rangle$. Computing the boundary maps, we have $\partial_0 u_0 = \partial_0 u_1 = \partial_0 u_2 = 0$ and $\partial_1(u_0, u_1) = u_1 - u_0$. Thus

$$H_0(K) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle u_0, u_1, u_2 \rangle}{\langle u_1 - u_0 \rangle} = \frac{\langle u_0, u_1 - u_0, u_2 \rangle}{\langle u_1 - u_0 \rangle} \cong \langle u_0, u_2 \rangle \cong \mathbb{Z}^2$$
$$H_1(K) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\{0\}}{\{0\}} = \{0\} .$$

For L, we have $C_0(L) = \langle w_0, w_1, w_2, w_3 \rangle$ and $C_1(L) = \langle (w_0, w_1), (w_2, w_3) \rangle$. Computing the boundary maps, we have $\partial_0 w_0 = \partial_0 w_1 = \partial_0 w_2 = \partial_0 w_3 = 0$, $\partial_1(w_0, w_1) = w_1 - w_0$, and $\partial_1(w_2, w_3) = w_3 - w_2$. Since $w_1 - w_0$ and $w_3 - w_2$ are linearly independent, we have

$$H_0(L) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle w_0, w_1, w_2, w_3 \rangle}{\langle w_1 - w_0, w_3 - w_2 \rangle} = \frac{\langle w_0, w_1 - w_0, w_2, w_3 - w_2 \rangle}{\langle w_1 - w_0, w_3 - w_2 \rangle} \cong \langle w_0, w_2 \rangle \cong \mathbb{Z}^2$$
$$H_1(L) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\{0\}}{\{0\}} = \{0\} .$$

So we have illustrated Theorem 5.12, as Rank $H_0(K) = 1$ and Rank $H_0(K) =$ Rank $H_0(L) = 2$ and we can see that J has a single path component while K and L each have two. We may use that $H_0(J) \ncong H_0(K)$ to conclude that $J \ncong K$, and similarly that $J \ncong L$. However, we cannot use homology to conclude whether or not K and L are homeomorphic since their sequence of homology groups is the same.

6 Cellular Homology

In the previous section we provided fairly minimal examples for computing the homology of simplicial complex. Our primary motivation for this is that simplicial homology computations quickly become unwieldy for spaces that are even slightly more complicated than those seen in the previous section, such as the 2-sphere or the torus. Thus, we introduce a new class of topological spaces whose structure allows us to more easily compute the homology groups. These topological spaces are called *cellular complexes* or *CW complexes*.

While it is generally easier to compute the homology of a space when viewed as a cellular complex rather than simplicial complex, it requires more algebraic topology than we have covered thus far to build a general cellular homology theory. Thus, we will state the general definition of a CW complex, as well as give a definition for the chain groups of CW complex, but we will refer the reader to [8] or [9] for a careful explanation of the boundary maps between chain groups.

6.1 CW Complexes

Definition 6.1. For a Hausdorff space X, an **(open)** n**-cell** e^n is an open set in X that is homeomorphic to $\mathring{B}^n$ [8].

Definition 6.2. Let X be a closed subset of a Hausdorff space X^* such that $X^* \setminus X$ contains an n-cell e^n. We say a continuous map $f : B^n \to \mathrm{Cl}(e^n)$ is a **characteristic map** if f maps $\mathring{B}^n$ homeomorphically onto e^n and $f(S^{n-1}) \subseteq X$ [8].

Definition 6.3. Let X be a closed subset of a Hausdorff space X^* such that $X^* \setminus X$ is the disjoint union of n-cells e^n_λ, for λ in some index set Λ where there is a characteristic map f_λ for each n-cell. We say that X^* has the **weak topology** induced by the maps f_λ and the inclusion map $X \to X^*$ provided that a subset A of X^* is closed if and only if $A \cap X$ and $f_\lambda^{-1}(A)$ are closed for each $\lambda \in \Lambda$. We may also think of X^* as being constructed from X by "pasting on" n-cells such that the characteristic map f_λ describes how each e^n_λ is pasted on [8].

Definition 6.4. A **CW complex** is a Hausdorff space X together with a structure prescribed by an ascending sequence of closed subspaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$$

satisfying:

(i) X^0 has the discrete topology.

(ii) For $n > 0$, X^n is constructed from X^{n-1} by adjoining a collection of n-cells e^n_λ such that the characteristic map f_λ describes how each e^n_λ is adjoined to X^n.

(iii) $X = \bigcup_i X^i$.

(iv) X and each X^i has the weak topology; that is, a subset A is closed if and only if $A \cap \mathrm{Cl}(e^n)$ is closed for all n-cells e^n, $n = 0, 1, 2, \ldots$.

We call each subset X^n the n**-skeleton** of X. Elements of X^0 are called **vertices** or **0-cells**. A CW complex is called **finite** if it contains finitely many cells, and **infinite** otherwise. If $X = X^n$ for some integer n, we say that the CW complex is **finite dimensional**, and the least such n is its **dimension** [8].

Definition 6.5. For cells e^m and e^n, we say that e^m is a **face** of e^n if $e^m \subseteq \mathrm{Cl}(e^n)$ and write $e^m \leq e^n$. If $e^m \neq e^n$, then we say e^m is a **proper face** of e^n [8].

Intuitively, to give a topological space X a CW complex structure is to think of building it out of (spaces homeomorphic to) open n-balls.

Example 6.6 (n-Sphere)**.** The n-sphere can be given a CW complex structure such that there are two cells: a vertex and an n-cell. In this case, X^k for $0 \leq k < n$ consists of some point $\mathbf{x} \in S^n$ and $X^n = S^n$. The characteristic map adjoining the n-cell e^n to $\mathbf{x}$ is, naturally, $f : B^n \to \mathrm{Cl}(e^n)$ where $f(S^{n-1}) = \mathbf{x}$ and $f(\mathring{B}^n)$ is the n-cell.

For $n = 2$, we may select our 0-cell to be $\mathbf{x} = (0, 0, 1)$ and then our 2-cell is $S^2 \smallsetminus \{\mathbf{x}\}$. Thus $X^0 = X^1 = \{\mathbf{x}\}$ and $X^2 = S^2$. The characteristic map $f : B^2 \to S^2$ adjoining our 2-cell $S^2 \smallsetminus \{\mathbf{x}\}$ to our 1-cell $\mathbf{x}$ is defined naturally by $f(S^1) = \mathbf{x}$ and $f(\mathring{B}^2) = S^2 \smallsetminus \{\mathbf{x}\}$.

Example 6.7 (Triangulable Spaces)**.** To triangulate a space is to give it a CW complex structure in which the (open) n-simplices are the n-cells.

We will see several more examples of giving familiar spaces a CW complex structure in the following section.

6.2 The Homology of CW Complexes

Let X be a topological space with a CW complex structure. Similar to simplicial homology, we can form a free abelian group generated by the n-cells of X. We denote this group $C_n(X)$ and call it the n**th chain group** of X. There is a boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ by which we can define the nth homology group of X to be $H_n(X) = \ker \partial_n / \operatorname{im} \partial_n$. This boundary map is dependent upon our choice of characteristic maps used to construct our space. As stated previously, defining the boundary map in general is outside of the scope of this paper. Instead, we choose to provide examples illustrating how to compute the homology of a cellular complex and the boundary maps will be clear from context. Note that we still have that ∂_0 is the zero map.

Theorem 6.8. *If X is a CW complex with no n-cells, then $H_n(X) = \{0\}$ [8].*

Example 6.9 (Cylinder)**.** We will build the cylinder as a quotient space from $I \times I$, and give it a CW complex structure as in Figure 11. Call this space X. The orientations of the edges A, B, and C imply their characteristic maps and boundaries.

Since there are two distinct vertices, we have $C_0(X) = \langle v, w \rangle \cong \mathbb{Z}^2$. By definition, $\partial_0 v = \partial_0 w = 0$, and so $\ker \partial_0 = \langle v, w \rangle \cong \mathbb{Z}^2$. Since there are three distinct edges, we have $C_1(X) = \langle A, B, C \rangle$. Then $\partial_1 A = v - w$ and $\partial_1 B = \partial_1 C = 0$. Thus $\operatorname{im} \partial_1 = \langle v - w \rangle$ and

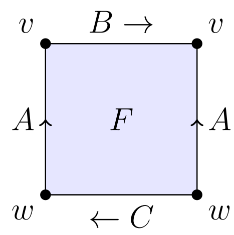


Figure 11: The cylinder with a CW complex structure.

$\ker \partial_1 = \langle B, C\rangle$. As with our simplicial homology computations, we will change our basis and use the First Isomorphism Theorem to find that

$$H_0(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle v, w\rangle}{\langle v - w\rangle} = \frac{\langle v - w, w\rangle}{\langle v - w\rangle} \cong \langle w\rangle \cong \mathbb{Z},$$

which indicates that the cylinder is connected. Now $C_2(X) = \langle F\rangle$, since the cylinder has one 2-cell. To compute the boundary of F, we think of traversing the edges of X clockwise (though counter-clockwise works just as well). From this we obtain a sum of the edges of X, where an edge is positive in the sum if we traverse the edge with its orientation, and negative if we traverse the edge against its orientation. This gives us $\partial_2 F = A + B - A + C = A - A + B + C = B + C$. Thus $\operatorname{im} \partial_2 = \langle B + C\rangle$ and $\ker \partial_2 = \{0\}$. Thus, similarly to above we have

$$H_1(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle B, C\rangle}{\langle B + C\rangle} = \frac{\langle B + C, C\rangle}{\langle B + C\rangle} \cong \langle C\rangle \cong \mathbb{Z},$$

indicating that there is a one-dimensional hole in the cylinder. Finally, since $\ker \partial_2 = \{0\}$ and since the cylinder has no n-cells for $n > 2$, $H_n(X) = \{0\}$ for all $n \geq 2$.

Example 6.10 (Projective Plane)**.** We construct the real projective plane as a quotient space from $I \times I$, give it a CW complex structure as in Figure 12, and call the resulting space X.

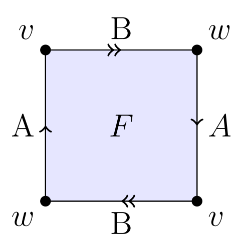


Figure 12: The real projective plane with a CW complex structure.

Our chain groups here are $C_0(X) = \langle v, w\rangle$, $C_1(X) = \langle A, B\rangle$, $C_2(X) = \langle F\rangle$, and $C_n(X) = \{0\}$ for $n > 2$. By definition $\ker \partial_0 = \langle v, w\rangle$. We have $\partial_1 A = v - w = -\partial_1 B$, so $\operatorname{im} \partial_1 =$

$\langle v - w \rangle$. Since the rank of $C_1(X)$ is two and the rank of $\operatorname{im} \partial_1$ is one, we know that the kernel of ∂_1 must have rank one, and so we must do some extra work to determine what elements are in $\ker \partial_1$. To do this, we will suppose that an element of $C_1(X)$ has no boundary and determine what properties it must have. Suppose that

$$0 = \partial_1(xA + yB) = x\partial_1 A + y\partial_1 B = x(v - w) + y(w - v) = (x - y)(w - v).$$

Then it must be that $x = y$. Thus, $\ker \partial_1 = \{x(A + B) \mid x \in \mathbb{Z}\} = \langle A + B \rangle$. Finally, $\partial_2 F = B + A + B + A = 2(A + B)$, and so $\operatorname{im} \partial_2 = \langle 2(A + B) \rangle$. Since the rank of $C_2(X)$ and $\operatorname{im} \partial_2$ are the same, we must conclude that $\ker \partial_2 = \{0\}$. Thus, the homology groups for the projective plane are

$$\begin{aligned} H_0(X) &= \frac{\langle v, w \rangle}{\langle v - w \rangle} \cong \langle w \rangle \cong \mathbb{Z}, \\ H_1(X) &= \frac{\langle A + B \rangle}{\langle 2(A + B) \rangle} \cong \mathbb{Z}_2, \\ H_n(X) &= \{0\} \quad \text{for } n \geq 2. \end{aligned}$$

Our calculation for the first homology group of the real projective plane is interesting, as this is the first time we have seen a homology group that is not either $\{0\}$ or some number of copies of $\mathbb{Z}$. We may interpret $H_1(X) \cong \mathbb{Z}_2$ as meaning that there is a loop bounding a 2-dimensional hole in the real projective plane that vanishes if we traverse it twice.

Example 6.11 (Torus)**.**

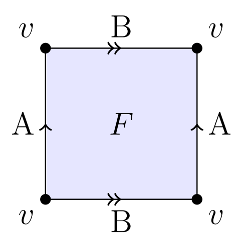


Figure 13: The torus with a CW complex structure.

We will use T to denote the torus constructed as a quotient space from $I \times I$ and given the CW complex structure in Figure 13. Our chain groups for T are $C_0(T) = \langle v \rangle$, $C_1(T) = \langle A, B \rangle$, $C_2(T) = \langle F \rangle$, and $C_n(T) = \{0\}$ for all $n > 2$. Computing the boundary maps, we have $\partial_0 v = 0$, $\partial_1 A = \partial_1 B = v - v = 0$, and $\partial_2 F = B - A - B + A = B - B + A - A = 0$.

Thus, the homology groups for the torus are

$$\begin{aligned}
H_0(T) &= \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle v \rangle}{\{0\}} = \langle v \rangle \cong \mathbb{Z}, \\
H_1(T) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle A, B \rangle}{\{0\}} = \langle A, B \rangle \cong \mathbb{Z}^2, \\
H_2(T) &= \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \frac{\langle F \rangle}{\{0\}} = \langle F \rangle \cong \mathbb{Z}, \\
H_n(T) &= \{0\} \quad \text{for } n > 2.
\end{aligned}$$

Example 6.12 (Klein bottle)**.**

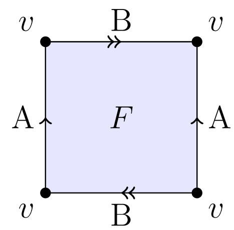


Figure 14: The Klein bottle with a CW complex structure.

Let K denote the Klein bottle as depicted in Figure 14. Its chain groups are $C_0(K) = \langle v \rangle$, $C_1(K) = \langle A, B \rangle$, $C_2(K) = \langle F \rangle$, and $C_n(K) = \{0\}$ for $n > 2$. Computing the boundary maps, we have $\partial_0 v = 0$, $\partial_1 A = \partial_1 B = v - v = 0$, and $\partial_2 F = B - A + B + A = 2B$. Thus, the homology groups for the Klein bottle are

$$\begin{aligned}
H_0(K) &= \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle v \rangle}{\{0\}} = \langle v \rangle \cong \mathbb{Z}, \\
H_1(K) &= \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle A, B \rangle}{\langle 2B \rangle} \cong \mathbb{Z} \times \mathbb{Z}_2, \\
H_2(K) &= \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \frac{\{0\}}{\{0\}} = \{0\}\,, \\
H_n(K) &= \{0\} \quad \text{for } n > 2.
\end{aligned}$$

The first homology group for the Klein bottle is interesting. We may interpret it as meaning that the Klein bottle has two 2-dimensional holes, but one of them behaves like the 2-dimensional hole in the real projective plane.

Let us compare the sequence of homology groups for the examples from this section.

$$\begin{aligned}
\text{Cylinder:} \quad & \mathbb{Z}, \mathbb{Z}, \{0\}, \{0\}, \{0\}, \ldots \\
\text{Real projective plane:} \quad & \mathbb{Z}, \mathbb{Z}_2, \{0\}, \{0\}, \{0\}, \ldots \\
\text{Torus:} \quad & \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}, \{0\}, \{0\}, \ldots \\
\text{Klein bottle:} \quad & \mathbb{Z}, \mathbb{Z} \times \mathbb{Z}_2, \{0\}, \{0\}, \{0\}, \ldots
\end{aligned}$$

Since no pair of these sequences is equal, we may conclude that none of these spaces are homeomorphic to each other.

7 Homology with Arbitrary Coefficients

In our study of homology theory thus far, we made a specific choice to define chain groups as $\mathbb{Z}$-modules. In fact, we can generalize homology theory so that our chain groups have coefficients from an arbitrary abelian group G. The generalization is natural, and we will state the generalization for CW complexes with the understanding that this also gives us a generalization for simplicial complexes.

Definition 7.1. Let X be a CW complex and G an abelian group. The **nth chain group of X with coefficients in G**, denoted $C_n(X;G)$, is the set of all linear combinations of n-cells in X with coefficients from G. If X has k n-cells, then $C_n(X,G) \cong G^k$ [9].

Definition 7.2. We define the **boundary map** $\partial_n : C(X;G) \to C_{n-1}(X;G)$ by the linear extension of the boundary of an n-cell; that is, for an n-cell σ and for $g \in G$

$$\partial_n(g\sigma) = g(\partial_n\sigma).$$

The nth homology group of X with coefficients in G is thus defined as

$$H_n(X;G) = \frac{\ker \partial_n}{\operatorname{im} \partial_n},$$

and is isomorphic to a k-fold direct product of G with itself [9].

Example 7.3. Let T denote the torus and K denote the Klein bottle. To compute the homology of T with $\mathbb{Z}_2$ coefficients is the same process as in Example 6.11, but our homology groups become

$$\begin{aligned}
&H_0(T;\mathbb{Z}_2) \cong \mathbb{Z}_2, \\
&H_1(T;\mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \\
&H_2(T;\mathbb{Z}_2) \cong \mathbb{Z}_2, \\
&H_n(T;\mathbb{Z}_2) = \{0\} \quad \text{for } n > 2.
\end{aligned}$$

Computing the homology of K with $\mathbb{Z}_2$ coefficients is nearly the same as in Example 6.12, except that we have $\partial_2 F = 2B \equiv 0 \pmod 2$. So our homology groups become

$$\begin{aligned}
&H_0(K;\mathbb{Z}_2) \cong \mathbb{Z}_2,\\
&H_1(K;\mathbb{Z}_2) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle A,B\rangle}{\{0\}} \cong \mathbb{Z}_2\times\mathbb{Z}_2,\\
&H_2(K;\mathbb{Z}_2) = \frac{\ker\partial_2}{\operatorname{im}\partial_3} = \frac{\langle F\rangle}{\{0\}} = \langle F\rangle \cong \mathbb{Z}_2,\\
&H_n(K;\mathbb{Z}_2) = \{0\} \quad \text{for } n>2.
\end{aligned}$$

Interestingly, homology with $\mathbb{Z}_2$ coefficients does not allow us to distinguish the torus from the Klein bottle whereas we saw that homology with $\mathbb{Z}$ coefficients does.

8 The Construction BG

We now introduce a new set of spaces.

Definition 8.1. Let G be an abelian group. We define

$$BG = \bigcup_{n\in\mathbb{Z}_{\geq 0}} (\Delta^n \times G^n)/\sim,$$

where $\Delta^n = \{(t_1,t_2,\dots,t_n)\in\mathbb{R}^n \mid 0\leq t_1\leq t_2\leq\cdots\leq t_n\leq 1\}$. We define the equivalence relation $\sim$ by

(1) If $g_i = e$, the identity in G, or if $t_i = t_{i+1}$, then we delete t_i and combine g_i and g_{i+1} via G's operation. Thus

$$(t_1,\dots,t_n,g_1,\dots,g_n) \sim (t_1,\dots,\hat{t}_i,\dots,t_n,g_1,\dots,\hat{g}_i,g_i\circ g_{i+1},\dots,g_n).$$

(2) If $t_1 = 0$ then t_1 and g_1 are deleted.

(3) If $t_n = 1$ then t_n and g_n are deleted [2].

The construction BG is an infinite cellular complex, where the cells are of the form $\Delta^n \times (G \setminus \{e\})^n$, for some $n\in\mathbb{Z}_{\geq 0}$. We can denote a cell by only its group elements using what we will call **bar notation**: $[g_1|g_2|\cdots|g_n]$. We can picture a cell as n symbols on the unit interval, where each element in $G\setminus\{e\}$ has its own symbol and the location of the symbol corresponding to g_i is the point t_i on the unit interval. We will call this representation the **particle model**.

Before we give examples of the construction BG, we need to solidify what we mean by dimension. There are two different dimensions we can define for a cell in BG, and we take the sum of those dimensions to be the actual dimension of the cell.

Definition 8.2. For $[g_1|\cdots|g_m] \in BG$, we define the **tensor dimension** of $[g_1|\cdots|g_m]$ by

$$d_t[g_1|\cdots|g_m] = d(g_1) + \cdots + d(g_m),$$

where $d(g_i)$ is the dimension of g_i in G. We define the **simplicial dimension** of $[g_1|\cdots|g_m]$ by

$$d_s[g_1|\cdots|g_m] = m.$$

Finally, we define the **total dimension** of $[g_1|\cdots|g_m]$ to be

$$d_B[g_1|\cdots|g_m] = d_t[g_1|\cdots|g_m] + d_s[g_1|\cdots|g_m]$$

[4].

Note that for groups such as $\mathbb{Z}$ or $\mathbb{Z}_p$, the dimension of an element is always taken to be 0. So for $B\mathbb{Z}_p$, an n-cell is what we might expect: n group elements, each with a corresponding time element. We will not need to worry about tensor dimension until Section 8.3. Throughout, we will use "n-cell" to refer to an element of BG with total dimension n.

Example 8.3. Let $G = \mathbb{Z}_2$. Since there is only one nonzero element in $\mathbb{Z}_2$, each n-cell in $B\mathbb{Z}_2$ is of the form $(t_1, t_2, \ldots, t_n, 1, 1, \ldots, 1)$. The following table illustrates how we may use bar notation and the particle model to represent cells.

Total dimension	Bar notation	Particle model
0	[]	——
1	[1]	—+—
2	[1|1]	—++—
3	[1|1|1]	—+++—
4	[1|1|1|1]	—++++—

Example 8.4. Let $G = \mathbb{Z}_3$. Since there are two nonzero elements in $\mathbb{Z}_3$, we have more variety in the form of our n-cells, which we can see in the following table.

Dimension	Cells
1	[1], [2]
2	[1|1], [1|2], [2|1], [2|2]
3	[1|1|1], [1|1|2], [1|2|1], [2|1|1], [1|2|2], [2|1|2], [2|2|1], [2|2|2]

The previous two examples suggest a theorem for the number of n-cells in $B\mathbb{Z}_p$.

Theorem 8.5. *The number of n-cells in $B\mathbb{Z}_p$ is $(p-1)^n$.*

Proof. Since we think of two n-cells $g, h \in B\mathbb{Z}_p$, where $g = [g_1|\cdots|g_n]$ and $h = [h_1|\cdots|h_n]$, as equal if $g_i = h_i$ for each i, the number of n-cells in $B\mathbb{Z}_p$ is also the cardinality of $(\mathbb{Z}_p \setminus \{0\})^n$. Since $(\mathbb{Z}_p \setminus \{0\})^n$ is the set of all n-tuples where there are $p-1$ choices for each entry, it follows that there are $(p-1)^n$ n-cells in $B\mathbb{Z}_p$. $\square$

There is an addition that we can define on BG, which we will denote with $*$.

Definition 8.6. We define an addition $* : BG \times BG \to BG$ such that for two cells $(t_1, \ldots, t_p, g_{t_1}, \ldots, g_{t_p})$ and $(s_1, \ldots, s_q, g_{s_1}, \ldots, g_{s_q})$ in BG

$$(t_1, \ldots, t_p, g_{t_1}, \ldots, g_{t_p}) * (s_1, \ldots, s_q, g_{s_1}, \ldots, g_{s_q}) = (\lambda_1, \ldots, \lambda_{p+q}, g_{\lambda_1}, \ldots, g_{\lambda_{p+q}}),$$

where the λ_i are the t_i and s_i arranged in increasing order [2].

Example 8.7. Let $G = \mathbb{Z}_3$. We have $(\frac{1}{10}, \frac{1}{5}, \frac{1}{3}, 1, 1, 1) * (\frac{1}{7}, \frac{1}{2}, 2, 2) = (\frac{1}{10}, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}, 1, 2, 1, 1, 2)$.

If we add two cells with a matching time component, we need to be a bit more careful. Adding $(\frac{1}{2}, \frac{2}{3}, 1, 2)$ and $(\frac{1}{3}, \frac{1}{2}, 2, 1)$ produces $(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, 2, 1, 1, 2)$. But by Definition 8.1, we have $(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}, 2, 1, 1, 2) = (\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 2, 2, 2)$.

8.1 The Homology of BG

We define the nth chain group of BG, $C_n(BG; G)$ to be the free abelian group generated by the n-cells of BG with coefficients in G. There are two boundary maps that we can define for cells in BG, and the map we use for computing homology is the sum of those two maps. Similarly with dimension, however, we will only be concerned with one of these until Section 8.3.

Definition 8.8. Let $[g_1|\cdots|g_m] \in BG$ be an n-cell. The **simplicial boundary** of $[g_1|\cdots|g_m]$ is the map $\partial_s : C_n(BG; G) \to C_{n-1}(BG; G)$ defined by

$$\begin{aligned}\partial_s[g_1|\cdots|g_m] = [g_2|\cdots|g_m]& \\ &+ \sum_{i=1}^{m-1} (-1)^{d_B[g_1|\cdots|g_i]}[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m] \\ &+ (-1)^n[g_1|\cdots|g_{m-1}],\end{aligned}$$

where by $[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m]$ we mean the element obtained by making $t_i = t_i + 1$. The **residual boundary** of $[g_1|\cdots|g_m]$ is the map $\partial_r : C_n(BG; G) \to C_{n-1}(BG; G)$ defined by

$$\partial_r[g_1|\cdots|g_m] = \sum_{i=1}^{m} (-1)^{d_B[g_1|\cdots|g_{i-1}]+1}[g_1|\cdots|\partial g_i|\cdots|g_m],$$

where ∂g_i is the boundary of g_i in G. Finally, we define the **total boundary** of $[g_1|\cdots|g_m]$ to be the map $\partial_n : C_n(BG; G) \to C_{n-1}(BG; G)$ defined by

$$\partial_n = \partial_s + \partial_r$$

[4].

Note that for groups such as $\mathbb{Z}_p$, elements have no boundary and so elements of $B\mathbb{Z}_p$ have no residual boundary. Thus, the total boundary of an element in $B\mathbb{Z}_p$ is its simplicial boundary. It also is important to be mindful that the boundary maps are computed with coefficients in G.

The simplicial boundary map has a nice geometric interpretation. Consider a cell $(t_1, \ldots, t_m, g_1, \ldots, g_m)$ in BG. The term $[g_2|\cdots|g_m]$ corresponds to t_1 being sent to 0, the term $[g_1|\cdots|g_{m-1}]$ corresponds to t_m being sent to 1, and the term $[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m]$ corresponds to t_i being sent to t_{i+1}.

Now we define the nth homology group of BG by $H_n(BG; G) = \ker \partial_n / \operatorname{im} \partial_{n+1}$.

8.2 The Homology of $B\mathbb{Z}_p$

There are some nice properties of the homology groups for $B\mathbb{Z}_p$ where p is prime. We will state them without proof for an arbitrary prime and explore them more in depth in the cases where $p = 2$ and $p = 3$.

Theorem 8.9. *For a prime p, $H_n(B\mathbb{Z}_p; \mathbb{Z}_p) \cong \mathbb{Z}_p$ [10].*

Thus, there is a unique generator of $H_n(B\mathbb{Z}_p; \mathbb{Z}_p)$ for any prime p. We will denote this generator by b_n. With this, we can define a nice addition on the collection of homology groups for $B\mathbb{Z}_p$ that is induced by the addition on $\mathbb{Z}_p$. One may think of adding two generators as "shuffling" them together.

Definition 8.10. We define an addition on the collection of all homology groups of $B\mathbb{Z}_p$

$$* : H_i(B\mathbb{Z}_p; \mathbb{Z}_p) \times H_j(B\mathbb{Z}_p; \mathbb{Z}_p) \to H_{i+j}(B\mathbb{Z}_p; \mathbb{Z}_p)$$

such that

$$b_i * b_j = b_j * b_i = \binom{i+j}{i} b_{i+j} \pmod{p}$$

[4].

It is clear from the definition that $b_i * b_j = 0$ only when $\binom{i+j}{i} \equiv 0 \pmod{p}$.

Theorem 8.11. *The generator b_n of $H_n(B\mathbb{Z}_p;\mathbb{Z}_p)$ can be expressed as sum of the generators b_{p^i} of $H_{p^i}(B\mathbb{Z}_p;\mathbb{Z}_p)$, which are determined by the base-p representation of n [10].*

8.2.1 The Homology of $B\mathbb{Z}_2$

Let us begin by computing the homology groups for $B\mathbb{Z}_2$. As noted in Section 8.1, the residual boundary map of elements in $B\mathbb{Z}_2$ is trivial, and so $\partial_n = \partial_s$ for each n. By Theorem 8.5, $C_n(B\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2$ for each n. Let $[g_1|g_2|\cdots|g_n]$, where $g_i = 1$ for each i, be the nontrivial n-cell in $C_n(B\mathbb{Z}_2;\mathbb{Z}_2)$. Computing it's boundary, we find

$$\begin{aligned}\partial_n[g_1|g_2|\cdots|g_n] &= \partial_s[g_1|g_2|\cdots|g_n] \\ &= [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}[g_1|\cdots|g_i+g_{i+1}|\cdots|g_n] + (-1)^n[g_1|g_2|\cdots|g_{n-1}].\end{aligned}$$

For each i, $g_i + g_{i+1} = 1 + 1 \equiv 0 \pmod{2}$. So $[g_1|\cdots|g_i+g_{i+1}|\cdots|g_n]$ is an $(n-2)$-cell in $B\mathbb{Z}_2$ by Definition 8.1 and thus is trivial in the boundary map as it is not in the codomain of ∂_n. Since each $g_i = 1$, we have

$$\begin{aligned}[g_2|\cdots|g_n] + (-1)^n[g_1|g_2|\cdots|g_{n-1}] &= [g_2|\cdots|g_n] + (-1)^n[g_2|\cdots|g_n] \\ &\equiv [g_2|\cdots|g_n] - [g_2|\cdots|g_n] \\ &\equiv 0 \pmod{2}.\end{aligned}$$

Thus ∂_n is the zero map, and so we have

$$H_n(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} = \frac{C_n(B\mathbb{Z}_2;\mathbb{Z}_2)}{\{0\}} \cong \mathbb{Z}_2$$

for each $n \geq 0$. We will use b_n to refer to the generator of $H_n(B\mathbb{Z}_2;\mathbb{Z}_2)$. Here b_n is also a generator for $C_n(B\mathbb{Z}_2;\mathbb{Z}_2)$.

In following example, we illustrate Theorem 8.11 to motivate our proof of the theorem for $p = 2$.

Example 8.12. We shall attempt by trial and error to find generators that can be expressed in terms of other generators to illustrate the pattern.

Certainly there is only one way to express b_1, and that is $b_1 = [1]$. We then ask ourselves if there is another way to express $b_2 = [1|1]$. The only candidate is $[1] * [1]$, but since

$$[1] * [1] = \binom{1+1}{1}[1|1] = \binom{2}{1}[1|1] \equiv 0 \pmod 2$$

we may conclude that the only way to express b_2 is $b_2 = [1|1]$.

Perhaps $b_3 = [1|1|1]$ will be more interesting. Since our addition is commutative, we have two candidates: $[1] * [1] * [1]$ and $[1] * [1|1]$. We have

$$[1] * [1] * [1] = \binom{3}{1,1,1}[1|1|1] = \binom{3}{1}\binom{2}{1}[1|1|1] \equiv 0 \pmod 2,$$

while

$$[1] * [1|1] = \binom{3}{1}[1|1|1] \equiv [1|1|1] \pmod 2.$$

Thus, we may write $b_3 = [1] * [1|1] = b_1 * b_2$.

As we proceed, notice that if we ever attempt to express b_n as a sum containing $[1] * [1]$, we will produce $\binom{2}{1}$ and so such an expression will never be a suitable replacement for b_n. The following table contains our above work as well as a few more attempts to express b_n as a sum of other generators so that the reader may find a pattern. For readability, we will write b_i in place of the bar notation.

n	Candidates	Conclusion
1	b_1	$b_1 = b_1$
2	$b_1 * b_1 = \binom{1+1}{1}b_2 = \binom{2}{1}b_2 \equiv 0 \pmod 2$	$b_2 = b_2$
3	$b_1 * b_2 = \binom{3}{1}b_3 \equiv b_3 \pmod 2$	$b_3 = b_1 * b_2$
4	$b_2 * b_2 = \binom{4}{2}b_4 = 6b_4 \equiv 0 \pmod 2$	$b_4 = b_4$
5	$b_1 * b_2 * b_2 = \binom{5}{1,2,2}b_5 = \binom{5}{1}\binom{4}{2}b_5 \equiv 0 \pmod 2$	
	$b_1 * b_4 = \binom{5}{1}b_5 \equiv b_5 \pmod 2$	$b_5 = b_1 * b_4$
6	$b_1 * b_2 * b_3 = \binom{6}{1,2,3}b_6 = \binom{6}{1}\binom{5}{2}b_6 \equiv 0 \pmod 2$	
	$b_2 * b_4 = \binom{6}{2}b_6 = 15b_6 \equiv b_6 \pmod 2$	$b_6 = b_2 * b_4$
7	$b_3 * b_4 = \binom{7}{3}b_7 = 35b_7 \equiv b_7 \pmod 2$	$b_7 = b_3 * b_4$
	$b_1 * b_2 * b_4 = \binom{7}{1,2,4}b_7 = \binom{7}{1}\binom{6}{2}b_7 \equiv b_7$	$b_7 = b_1 * b_2 * b_4$
8	$b_2 * b_3 * b_3 = \binom{8}{2,3,3}b_8 \equiv 0 \pmod 2$	
	$b_4 * b_4 = \binom{8}{4}b_8 \equiv 0 \pmod 2$	$b_8 = b_8$

It appears that we can express each b_n as a sum of generators of the form b_{2^i}, where the sum is determined by the base-2 representation of n. This is in fact the case, as we will show. First, however, we need to introduce Kummer's Theorem.

Theorem 8.13 (Kummer's Theorem)**.** *For a prime number p, the largest power of p that divides $\binom{n}{k}$ is the number of carries when adding k and $n-k$ in base-p [7].*

Theorem 8.14. *The generator b_n of $H_n(B\mathbb{Z}_2;\mathbb{Z}_2)$ can be expressed as sum of the generators b_{2^i} of $H_{2^i}(B\mathbb{Z}_2;\mathbb{Z}_2)$, which are determined by the base-2 representation of n.*

Proof. It is well known that every nonnegative integer has a unique base-2 representation, and so we will not prove it here. Consider the binomial coefficient $\binom{2^i+2^j}{2^i}$. By Kummer's Theorem, $\binom{2^i+2^j}{2^i} \equiv 0 \pmod 2$ if and only if there is at least one carry when adding 2^i and 2^j in base-2, which occurs if and only if $i = j$, since we are adding powers of two. Thus if $i \neq j$, then we may write $b_{2^i+2^j} = b_{2^i} * b_{2^j}$, and if $i = j$, then we keep the original representation $b_{2^{i+1}}$.

We will now extend our argument to any finite sum of powers of 2. Suppose that $n = \sum_{i=0}^{k} c_i 2^i$, where c_i is either 1 or 0 for each i. We would like to express b_n as

$$\sum_{\substack{i=0\\c_i\neq 0}}^{k} b_{2^i} = \binom{n}{c_0 2^0, c_1 2^1, \dots, c_k 2^k} b_n = \binom{n}{c_0 2^0}\binom{\sum_{i=1}^{k} c_i 2^i}{c_1 2^1}\cdots\binom{c_{k-1}2^{k-1}+c_k 2^k}{c_{k-1}2^{k-1}} b_n,$$

where we abuse the symbol $\sum$ and use it to refer to both addition on the homology groups of $B\mathbb{Z}_2$ and addition in $\mathbb{R}$. To know if this is a valid representation of b_n, we need to verify that two does not divide

$$\binom{\sum_{i=\ell}^{k} c_i 2^i}{c_\ell 2^\ell}$$

for each ℓ, $0 \leq \ell \leq k$, such that $c_\ell \neq 0$. Certainly this is true, as we are adding 2^ℓ and $\sum_{i=\ell+1}^{k} c_i 2^i$ in base-2. Since the base-2 representation of $\sum_{i=\ell+1}^{k} c_i 2^i$ has a zero in its ℓth position and the base-2 representation of 2^ℓ has a one only in its ℓth position, we produce no carries when adding these numbers in base 2. Thus, we may write

$$b_n = \sum_{\substack{i=0\\c_i\neq 0}}^{k} b_{2^i}.$$

$\square$

8.2.2 The Homology of $B\mathbb{Z}_3$

Theorem 8.15. *The generator b_n of $H_n(B\mathbb{Z}_3;\mathbb{Z}_3)$ can be expressed as sum of the generators b_{3^i} of $H_{3^i}(B\mathbb{Z}_3;\mathbb{Z}_3)$, which are determined by the base-3 representation of n.*

Proof. For each $n \geq 0$, $H_n(B\mathbb{Z}_3;\mathbb{Z}_3) \cong \mathbb{Z}_3$. Any generator of $C_n(B\mathbb{Z}_3;\mathbb{Z}_3)$ is a suitable generator for $H_n(B\mathbb{Z}_3;\mathbb{Z}_3)$, so fix a generator b_n of $C_n(B\mathbb{Z}_3;\mathbb{Z}_3)$ and let $H_n(B\mathbb{Z}_3;\mathbb{Z}_3) = \langle b_n\rangle$. First, observe that since $\mathbb{Z}_3$ is a field, we have $H_n(B\mathbb{Z}_3;\mathbb{Z}_3) = \langle b_n\rangle = \langle 2b_n\rangle$.

Now, consider $b_{3^i} * b_{3^i} * b_{3^i} = \binom{3^i+3^i+3^i}{3^i} b_{3^{i+1}}$ for some $i \geq 0$. We will show that this is trivial using Kummer's Theorem. The base-3 representation of 3^i has a 1 in its ith position, and the base-3 representation of $3^i + 3^i$ has a 2 in its ith position. So when we add 3^i and $3^i + 3^i$ in base-3, we produce a carry. Thus, by Kummer's Theorem, $b_{3^i} * b_{3^i} * b_{3^i} = \binom{3^i+3^i+3^i}{3^i} b_{3^{i+1}} \equiv 0 \pmod 3$. From this we may conclude that, for each i, $H_{3^i}(B\mathbb{Z}_3) = \langle b_{3^i}\rangle$; that is, the only way to express b_{3^i} as a sum of generators of the form b_{3^j} is to express it as b_{3^i}.

Consider $b_{3^i} * b_{3^j} = \binom{3^i+3^j}{3^i} b_{3^i+3^j}$. To evaluate $\binom{3^i+3^j}{3^i} \pmod 3$, we will use Kummer's Theorem. The base-3 representations of 3^i and 3^j are a 1 in the ith position and jth position, respectively, with zeros elsewhere. Adding 3^i and 3^j in their base-3 representations produces an 1 in the ith and jth position if $i \neq j$, or a 2 in the ith position if $i = j$, with zeros elsewhere. Neither of these additions produce a carry in base-3, and so $\binom{3^i+3^j}{3^i} \not\equiv 0 \pmod 3$. Thus $b_{3^i} * b_{3^j}$ is a suitable replacement for $b_{3^i+3^j}$.

Suppose $n = \sum_{i=0}^{k} c_i 3^i$, where $c_i \in \{0, 1, 2\}$. Then, for all $0 \leq m \leq k$, $\binom{n}{3^m} \not\equiv 0 \pmod 3$, since the base-3 representation of n has at most a 2 in its mth position, and so $n - 3^m$ has a most a 1 in its mth position. Thus there are no carries when adding 3^m and $n-3^m$ in base-3. Thus, if we take $2b_i$ to mean $b_i * b_i$ and use $\sum$ to mean the addition on the generators of $H_i(B\mathbb{Z}_3;\mathbb{Z}_3)$, we have that $\sum_{i=0}^{k} c_i b_{3^i}$ is nontrivial, and so we may let $b_n = \sum_{i=0}^{k} c_i b_{3^i}$ □

8.3 Iterating

In Definition 8.1 the only restriction on G is that it is additive abelian. Our addition defined on BG in Definition 8.6 is commutative, and so BG is an additive abelian group. Thus we may iterate this construction so that

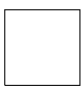
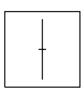
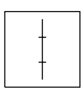
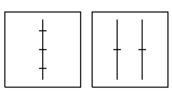
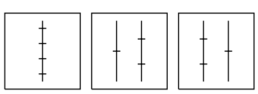
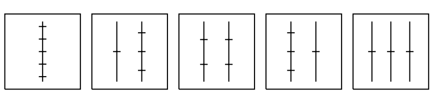
$$B(BG) = B^2G = \bigcup_{n\in\mathbb{Z}_{\geq 0}} (\Delta^n \times (BG)^n)/\sim,$$

where $\sim$ is defined as in Definition 8.1. The bar construction extends naturally to B^2G, and the particle model for elements of B^2G is elements from BG in the unit square. The dimension of an element in B^2G is its simplicial dimension plus the simplicial dimension of each of its group elements, as specified in Definition 8.2. In general, an element in B^2G is of the form $(s_1, \dots, s_m, G_1, \dots, G_m)$, where each G_m is of the form $(t_1, \dots, t_k, g_1, \dots, g_k)$. Though we will not explore it here, it is possible to iterate this construction n times so that $B(B^{n-1}G) = B^nG$.

Theorem 8.16. *There are no* 1*-cells in* B^2G*.*

Proof. We will argue this by contradiction. Suppose that we have a 1-cell. Such a cell would necessarily be of the form $[g]$, where the total dimension of g in BG is 0, as this would give us simplicial dimension 1 and tensor dimension 0, and so our cell would have total dimension 1. However, the only element in BG that has total dimension 0 is [], which is the identity in BG. By Definition 8.1, our cell is equivalent to a 0-cell in B^2G. Thus, we have our contradiction, and so there are no 1-cells in B^2G. □

Example 8.17 (Cells in $B^2\mathbb{Z}_2$)**.** In the following table, we give the particle model representation for all n-cells in $B^2\mathbb{Z}_2$, for $0 \leq n \leq 6$.

n	n-cells	Number of n-cells
0		1
1	$\emptyset$	0
2		1
3		1
4		2
5		3
6		5

From the previous example, it appears that the number of n-cells in $B^2\mathbb{Z}_2$ is a Fibonacci number. This is in fact the case, as we prove in the following theorem.

Theorem 8.18. *The number of n-cells in $B^2\mathbb{Z}_2$ is F_{n-1}, where F_n is the nth Fibonacci number, with $F_0 = 0$ and $F_1 = 1$.*

Proof. Let f_n be the number of n-cells in $B^2\mathbb{Z}_2$. Given a particular n-cell, look at the rightmost element in the particle model representation of the cell. If the rightmost element is a line with two or more dashes, removing a dash produces an $(n-1)$-cell in $B^2\mathbb{Z}_2$, of which there are f_{n-1}. If the rightmost element is a line with only one dash, however, removing that dash also requires us to remove the line. This is because an empty line represents the identity element in $B\mathbb{Z}_2$, and, by Definition 8.1, if any group element is the identity we delete that element. Thus, if the rightmost element is a line with a single dash, removing that dash produces an $(n-2)$-cell in $B^2\mathbb{Z}_2$, of which there are f_{n-2}. Summing over all possibilities, we find that $f_n = f_{n-1} + f_{n-2}$. Our initial terms are $f_1 = 0$ and $f_2 = 1$, and so our indices are shifted from the traditional Fibonacci numbers. With an index shift, we have our desired result: the number of n-cells in $B^2\mathbb{Z}_2$ is $f_n = F_{n-1}$. □

A common combinatorial interpretation of the Fibonacci numbers is that F_{n+1} is the number of tilings of a $1 \times n$ board with squares and dominoes. In our proof of the previous theorem, we may think of a line with a single dash as a "domino" and any additional dashes after the first as "squares".

There is a natural extension of our proof for the previous theorem to counting the number of n-cells in $B^2\mathbb{Z}_p$ for any prime p.

Theorem 8.19. *Let p be prime. Let S_n be the number of n-cells in $B^2\mathbb{Z}_p$. Then S_n is given by the recursion $(p-1)S_{n-1} + (p-1)S_{n-2}$ with initial conditions $S_1 = 0$ and $S_2 = p-1$.*

Proof. Let S_n denote the number of n-cells in $B^2\mathbb{Z}_p$. Given a particular n-cell, look at the rightmost element in the particle model representation of that cell. If the rightmost element in the cell is a line with 2 or more symbols, removing one produces an $(n-1)$-cell in $B^2\mathbb{Z}_p$, of which there are S_{n-1}. However, there are $p-1$ different symbols that we could have removed, one for each of the $p-1$ nonzero elements in $\mathbb{Z}_p$. Thus, there are $(p-1)S_{n-1}$ n-cells in $B^2\mathbb{Z}_p$ whose rightmost element is a line with two or more symbols. If instead the rightmost element in the n-cell is a line with one symbol, there are $p-1$ possible symbols it could be and removing produces an $(n-2)$-cell in $B^2\mathbb{Z}_p$, of which there are S_{n-2}. Thus, there are $(p-1)S_{n-2}$ n-cells in $B^2\mathbb{Z}_p$ whose rightmost element is a line with a single symbol. Summing over all possibilities, we find that the number of n-cells in $B^2\mathbb{Z}_p$ is $S_n = (p-1)S_{n-1} + (p-1)S_{n-2}$. By Theorem 8.16, $S_1 = 0$. By Definition 8.1, a 2-cell in $B^2\mathbb{Z}_p$ necessarily has simplicial dimension 1. By Definition 8.2, a 2-cell in $B^2\mathbb{Z}_p$ must then have tensor dimension 1. Thus, all 2-cells in $B^2\mathbb{Z}_p$ are of the form $[g]$ where g is a 1-cell in $B\mathbb{Z}_p$. By Theorem 8.5, there are $p-1$ 1-cells in $B\mathbb{Z}_p$. Thus, $S_2 = p-1$. □

8.4 The Homology of $B^2\mathbb{Z}_p$

Now that we know how many n-cells there are in $B^2\mathbb{Z}_p$, we may ask about the homology of $B^2\mathbb{Z}_p$, which are computed with $\mathbb{Z}_p$ coefficients. Let us begin by analyzing the boundary maps for $B^2\mathbb{Z}_p$. Let $[g_1|\cdots|g_m]$ be an n-cell in $B^2\mathbb{Z}_p$. From Definition 8.8, the simplicial boundary map is

$$\begin{aligned}\partial_s[g_1|\cdots|g_m] = [g_2|\cdots|g_m]\\ &+ \sum_{i=1}^{m-1}(-1)^{d_B[g_1|\cdots|g_i]}[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m]\\ &+ (-1)^n[g_1|\cdots|g_{m-1}].\end{aligned}$$

However, the maximum total dimension of $[g_2|\cdots|g_m]$ is $n-1-d(g_m)$. Since the minimum dimension of g_1 is 1, the maximum total dimension of $[g_2|\cdots|g_m]$ is $n-2$. Thus $[g_2|\cdots|g_m]$ is trivial in the boundary map as it is not of the correct dimension. There is a similar argument for $[g_1|\cdots|g_{m-1}]$. Thus, the simplicial boundary map for $B^2\mathbb{Z}_p$ becomes

$$\partial_s[g_1|\cdots|g_m] = \sum_{i=1}^{m-1}(-1)^{d_B[g_1|\cdots|g_i]}[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m].$$

From Definition 8.8, the residual boundary map is

$$\partial_r[g_1|\cdots|g_m] = \sum_{i=1}^{m}(-1)^{d_B[g_1|\cdots|g_{i-1}]+1}[g_1|\cdots|\partial g_i|\cdots|g_m].$$

In general, this does not drop out, and so we take the total boundary of $[g_1|\cdots|g_m]$ to be

$$\begin{aligned}\partial_n[g_1|\cdots|g_m] &= \sum_{i=1}^{m-1}(-1)^{d_B[g_1|\cdots|g_i]}[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m]\\ &\quad+ \sum_{i=1}^{m}(-1)^{d_B[g_1|\cdots|g_{i-1}]+1}[g_1|\cdots|\partial g_i|\cdots|g_m].\end{aligned}$$

Let us now restrict our attention to $B^2\mathbb{Z}_2$. Let $[g_1|\cdots|g_m]$ be an n-cell in $B^2\mathbb{Z}_2$. Our first observation is that the residual boundary map is trivial. In Section 8.2.1, we found that the boundary map $\partial : C_k(B\mathbb{Z}_2;\mathbb{Z}_2) \to C_{k-1}(B\mathbb{Z}_2;\mathbb{Z}_2)$ was trivial for all $k \geq 0$. Thus, in the residual boundary map, the term $[g_1|\cdots|\partial g_i|\cdots|g_m]$ becomes $[g_1|\cdots|[]|\cdots|g_m]$, which becomes $[g_1|\cdots|\hat{g}_i|\cdots|g_m]$ by Definition 8.1. By a similar argument as above, the total dimension of $[g_1|\cdots|\hat{g}_i|\cdots|g_m]$ must be less than $n-1$. Since this holds for all i, $1 \leq i \leq m$, we have that the residual boundary map $\partial_r : C_n(B\mathbb{Z}_2;\mathbb{Z}_2) \to C_{n-1}(B\mathbb{Z}_2;\mathbb{Z}_2)$ is the zero map.

Thus, the total boundary map in $B^2\mathbb{Z}_2$ is

$$\partial_n[g_1|\cdots|g_m] = \sum_{i=1}^{m-1}[g_1|\cdots|g_i \circ g_{i+1}|\cdots|g_m],$$

since $1 \equiv -1 \pmod 2$.

Now, from Theorem 8.18, we can think of an n-cell $[g_1|\cdots|g_m]$ in $B^2\mathbb{Z}_2$ as constructed from either an $(n-1)$-cell or an $(n-2)$-cell. Explicitly, if $g_m = [1]$, then $[g_1|\cdots|g_m]$ was obtained from an $(n-2)$-cell $[g_1|\cdots|g_{m-1}]$ by appending $[1]$. If $d_s(g_m) \geq 2$, then $[g_1|\cdots|g_m]$ was obtained from an $(n-1)$-cell $[g_1|\cdots|h_m]$, where h_m has $d_s(h_m) = d_s(g_m) - 1$. We can use this recursive construction of n-cells to explore the homology of $B^2\mathbb{Z}_2$.

Let $g = [g_1|\cdots|g_m]$ be an n-cell in $B^2\mathbb{Z}_2$. If g is constructed from an $(n-1)$-cell, we will use $\dot{g}$ to refer to that $(n-1)$-cell. If g is constructed from an $(n-2)$-cell, we will use $\ddot{g}$ to refer to that $(n-2)$-cell. We will use *predecessor* to refer to $\dot{g}$ or $\ddot{g}$, according to the cell that g is constructed from. We will use $\dot{g}_i$ to refer to the ith element of $\dot{g}$ and $\ddot{g}_i$ to refer to the ith element of $\ddot{g}$. For simplicity, we will use b_n to refer to the generator of $H_n(B\mathbb{Z}_2;\mathbb{Z}_2)$; that is, the cell whose bar notation consists of n 1's. From our above discussion we have that the boundary of g is

$$\partial_n g = \sum_{i=1}^{m-1}[g_1|\cdots|g_i * g_{i+1}|\cdots|g_m] = \sum_{i=1}^{m-1}\binom{d(g_i)+d(g_{i+1})}{d(g_i)}[g_1|\cdots|b_{d(g_i)+d(g_{i+1})}|\cdots|g_m].$$

We will break this into cases depending on the predecessor of g.

If g is constructed from an $(n-2)$-cell, notice that $d_s\ddot{g} = m-1$. Thus

$$\begin{aligned}\partial_n g = &\sum_{i=1}^{m-2}\binom{d(\ddot{g}_i)+d(\ddot{g}_{i+1})}{d(\ddot{g}_i)}[\ddot{g}_1|\cdots|b_{d(\ddot{g}_i)+d(\ddot{g}_{i+1})}|\cdots|\ddot{g}_{m-1}|b_1] \\ &+\binom{d(\ddot{g}_{m-1})+1}{1}[\ddot{g}_1|\cdots|\ddot{g}_{m-2}|b_{d(\ddot{g}_{m-1})+1}].\end{aligned}$$

That is, the first $m-2$ terms of $\partial_n g$ are precisely the same as for $\partial_{n-2}\ddot{g}$, but with b_1 appended to the end of each term. If $d(\ddot{g}_{m-1})$ is odd, then $\binom{d(\ddot{g}_{m-1})+1}{1} \equiv 0 \pmod 2$, and so

$$\partial_n g = \sum_{i=1}^{m-2}\binom{d(\ddot{g}_i)+d(\ddot{g}_{i+1})}{d(\ddot{g}_i)}[\ddot{g}_1|\cdots|b_{d(\ddot{g}_i)+d(\ddot{g}_{i+1})}|\cdots|\ddot{g}_{m-1}|b_1].$$

If $d(\ddot{g}_{m-1})$ is even, then $\binom{d(\ddot{g}_{m-1})+1}{1} \equiv 1 \pmod 2$, and so

$$\partial_n g = \sum_{i=1}^{m-2} \binom{d(\ddot{g}_i) + d(\ddot{g}_{i+1})}{d(\ddot{g}_i)} [\ddot{g}_1|\cdots|b_{d(\ddot{g}_i)+d(\ddot{g}_{i+1})}|\cdots|\ddot{g}_{m-1}|b_1] + [\ddot{g}_1|\cdots|\ddot{g}_{m-2}|b_{d(\ddot{g}_{m-1})+1}].$$

Suppose now that g is constructed from an $(n-1)$-cell. Notice that $d_s g = d_s \dot{g} = m$. We have

$$\begin{aligned}\partial_n g = &\sum_{i=1}^{m-2} \binom{d(\dot{g}_i) + d(\dot{g}_{i+1})}{d(\dot{g}_i)} [\dot{g}_1|\cdots|b_{d(\dot{g}_i)+d(\dot{g}_{i+1})}|\cdots|\dot{g}_{m-1}|g_m] \\ &+ \binom{d(\dot{g}_{m-1}) + d(\dot{g}_m) + 1}{d(\dot{g}_{m-1})} [\dot{g}_1|\cdots|\dot{g}_{m-2}|b_{d(\dot{g}_{m-1})+d(\dot{g}_m)+1}].\end{aligned}$$

That is, the boundary of g is the first $m-2$ terms of the boundary of $\dot{g}$, where the dimension of $\dot{g}_m$ is increased by one, plus the final term of the boundary of $\dot{g}$ where the top number in the binomial coefficient has been increased by 1 and the dimension of the last entry has been increased by 1.

Since only the final binomial coefficient changes from $\partial_{n-1}\dot{g}$, we need only compute whether $\binom{d(\dot{g}_{m-1})+d(\dot{g}_m)+1}{d(\dot{g}_{m-1})}$ is even or odd. This problem is equivalent to determining a pattern for the parity of the entries in a given column of Pascal's triangle – a problem left unsolved at the conclusion of this project.

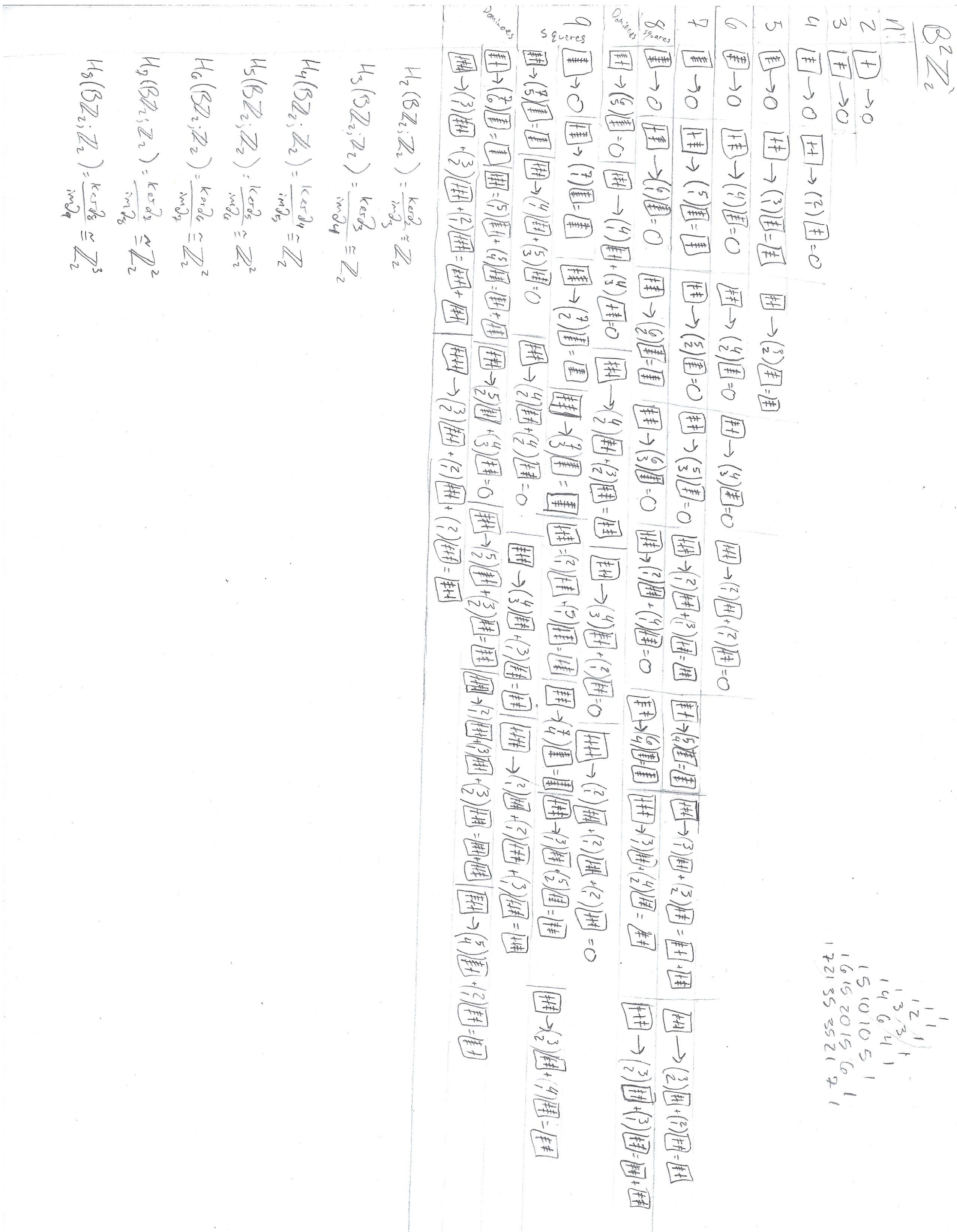
We leave the reader with a conjecture.

Conjecture. *If n is equal to $3k-1$, $3k$, or $3k+1$ for some $k \in \mathbb{Z}_+$, then*

$$H_n(B\mathbb{Z}_2; \mathbb{Z}_2) \cong \mathbb{Z}_2^k.$$

In the appendix we provide work to support this conjecture and show that it holds for $2 \leq n \leq 8$.

Appendix


$B^2\mathbb{Z}_2$
Squares
Dominoes
$H_2(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker\partial_2}{\operatorname{im}\partial_3} \cong \mathbb{Z}_2$
$H_3(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker\partial_3}{\operatorname{im}\partial_4} \cong \mathbb{Z}_2$
$H_4(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker\partial_4}{\operatorname{im}\partial_5} \cong \mathbb{Z}_2$
$H_5(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker\partial_5}{\operatorname{im}\partial_6} \cong \mathbb{Z}_2^2$
$H_6(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker\partial_6}{\operatorname{im}\partial_7} \cong \mathbb{Z}_2^2$
$H_7(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker\partial_7}{\operatorname{im}\partial_8} \cong \mathbb{Z}_2^2$
$H_8(B\mathbb{Z}_2;\mathbb{Z}_2) = \frac{\ker\partial_8}{\operatorname{im}\partial_9} \cong \mathbb{Z}_2^3$

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