Math 2002 Number Systems
Homework Set 6

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Course Instructor: Dr. Markus Pflaum

Contact Info: Office: Math 255, Telephone: 2-7717, e-mail: `markus.pflaum@colorado.edu`.

For the following problems recall that the set $\mathbb{Q}$ of rational numbers is defined as the quotient set $\mathbb{Q} = (\mathbb{Z} \times \mathbb{Z}^*)/\sim$, where $\sim$ is the equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$ defined as follows:

$$(p, q) \sim (\tilde{p}, \tilde{q}) \iff p \cdot \tilde{q} = \tilde{p} \cdot q \quad \text{where } p, \tilde{p} \in \mathbb{Z},\ q, \tilde{q} \in \mathbb{Z}^* \ .$$

Recall further that $\frac{p}{q}$ denotes the equivalence class of the pair (p, q). Addition on $\mathbb{Q}$ is then defined by

$$+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q},\ \left(\frac{p}{q}, \frac{k}{l}\right) \mapsto \frac{pl + kq}{ql} \ ,$$

and multiplication by

$$\cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q},\ \left(\frac{p}{q}, \frac{k}{l}\right) \mapsto \frac{p \cdot k}{q \cdot l} \ .$$

The set $\mathbb{Z}$ of integers is embedded into $\mathbb{Q}$ via the map $\mathbb{Z} \hookrightarrow \mathbb{Q}$, $p \mapsto \frac{p}{1}$.

Problem 1: Show that both addition and multiplication on $\mathbb{Q}$ are well-defined. (3P)

Problem 2: Verify the following properties of addition and multiplication in $\mathbb{Q}$:

(a) associativity of addition, (2P)

(b) commutativity of addition, (1P)

(c) additive neutrality of $0 = \frac{0}{1}$, (1P)

(d) existence of additive inverses, (1P)

(e) associativity of multiplication, (2P)

(f) commutativity of multiplication, (1P)

(g) multiplicative neutrality of $1 = \frac{1}{1}$, (1P)

(h) existence of multiplicative inverses for $\frac{p}{q} \neq 0$, (2P)

(i) distributivity of multiplication over addition. (2P)

Problem 3: Define an order relation on $\mathbb{Q}$ as follows:

$$\frac{k}{l} \leq \frac{p}{q} \iff (pl - kq) \cdot ql \geq 0 \ .$$

Verify that $\leq$ is welldefined, an order relation on $\mathbb{Q}$ indeed and that it satisfies the following monotony laws, where r, s are always rational numbers.

Monotony of addition
If $r \leq s$ and $a \in \mathbb{Q}$, then $r + a \leq s + a$.

Monotony of multiplication
If $r \leq s$ and $a \in \mathbb{Q}$ with $a \geq 0$, then $r \cdot a \leq s \cdot a$.

(8P)