Math 3001 Analysis 1
Homework Set 7

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Theorem 1 (Inverse Function Theorem). *Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ a differentiable function such that $f'(x_0) \neq 0$ for some $x_0 \in I$. Then there exist open intervals $I_0 \subset I$ and $J_0 \subset \mathbb{R}$ with $x_0 \in I_0$ such that $f(I_0) = J_0$, $f'(x) \neq 0$ for all $x \in I_0$ and such that the restriction $f|_{I_0} : I_0 \to J_0$ is invertible with differentiable inverse function $g : J_0 \to I_0$. The derivative of the inverse function is given by*

$$g' : J_0 \to \mathbb{R}, \quad y \mapsto g'(y) = \frac{1}{f'(g(y))} \ .$$

Problem 1 Show that the exponential function $\exp : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^x$ is strictly monotone increasing, and maps $\mathbb{R}$ onto $\mathbb{R}_{>0}$. Let $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ be the corresponding inverse function. Compute the derivative of ln by using the Inverse Function Theorem (4P)

Problem 2: Recall that $\cosh x := \frac{e^x+e^{-x}}{2}$ and $\sinh x := \frac{e^x-e^{-x}}{2}$. Prove that the function $\tanh : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{\sinh x}{\cosh x}$ is strictly increasing and everywhere differentiable. Compute the derivative $\tanh'$ and the image I of tanh. Then determine the derivative of the inverse function $\mathrm{Artanh} : I \to \mathbb{R}$. (5P)

Problem 3: Compute the following integrals:

a)

$$\int \frac{1}{\sin(2t)} dt,$$

b)

$$\int \frac{1}{t^4 - 16} dt.$$

(4P)

Problem 4: Determine the integral

$$\int \frac{1}{ax^2 + bx + c} \, dx$$

depending on $a, b, c \in \mathbb{R}$, where $a \neq 0$ is assumed. (3P)

Problem 5: Determine the following integrals:

a)

$$\int_0^{2\pi} x \cos x \, dx,$$

b)

$$\int_0^{\pi} x \sin x \, dx.$$

(4P)