Math 3001 Analysis 1
Homework Set 6
Spring 2021

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Problem 1: Let $f : [a, b] \to [a, b]$ with $a < b$ be a continuous function. Prove that f has a fixed point, i.e. that there is an $x_0 \in [a, b]$ such that $f(x_0) = x_0$.
Hint: Use the Intermediate Value Theorem. (4P)

Problem 2: Prove that the function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is $\mathcal{C}^\infty$, and determine all derivatives $f^{(k)}(0)$, $k \in \mathbb{N}$.
Hint: Use Problem 3 from Homework 5. (4P)

Problem 3: Let $f : I \to \mathbb{R}$ be a function defined on an open interval I. Show that f being differentiable at $a \in I$ is equivalent to the existence of a function $E : I \to \mathbb{R}$ continuous at a such that

$$f(x) = f(a) + f'(a)(x - a) + E(x)(x - a) \quad \text{for all } x \in I$$

and $E(a) = 0$. (4P)

Problem 4: Determine the derivatives of the following functions on their maximal real domains:

a) $f(x) = \dfrac{x^2 - 5x + 6}{x^2 - 3x + 2}$, b) $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$,

c) $f(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$, d) $f(x) = \sqrt{|x|^3}$.

(8P)

Extra Credit Problem: Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$ denote the *extended real line*. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the extended real line and define the *limit inferior* of the sequence $(x_n)_{n\in\mathbb{N}}$ by

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} x_m$$

and its *limit superior* by

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m \ .$$

Show that the limit inferior and the limit superior always exist in $\overline{\mathbb{R}}$ and that

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \ .$$

Then prove that $(x_n)_{n\in\mathbb{N}}$ converges in $\overline{\mathbb{R}}$ if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n \ .$$

(8P)