Math 3001 Analysis 1
Homework Set 4

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Extra Problem: Let $\sum_{n=0}^{\infty} v_n$ and $\sum_{n=0}^{\infty} w_n$ be two series of complex numbers. Define their so-called *Cauchy product* as the series $\sum_{n=0}^{\infty} z_n$ defined by

$$z_n := \sum_{k=0}^{n} v_k \, w_{n-k} \, .$$

Prove that if both $\sum_{n=0}^{\infty} v_n$ and $\sum_{n=0}^{\infty} w_n$ are absolutely convergent, then their Cauchy product $\sum_{n=0}^{\infty} z_n$ is absolutely convergent as well, and

$$\Big(\sum_{n=0}^{\infty} v_n \Big) \cdot \Big(\sum_{n=0}^{\infty} w_n \Big) = \sum_{n=0}^{\infty} z_n \, .$$

(6 EP)

Problem 1: Show that for each natural $k \geq 2$ the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ converges.

Hint: You can use the result proven in class that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. (4P)

Problem 2: Consider the following two sequences and determine whether they converge or not. In the convergent case determine the limit.

$$\text{(a)} \quad x_n = \frac{13n^3 + 5n^2 + 3n + 8}{13 + 5n + 3n^2 + 8n^3}, \qquad \text{(b)} \quad y_n = \frac{2^n}{(n+2)!}.$$

Hint: The proof of the ratio test from class might help for the solution of Problem (b)! (4P)

Problem 3: Use the comparison, root or ratio test to determine whether the following series converge or diverge:

$$\text{a)} \quad \sum_{n=0}^{\infty} \frac{2^n n^3}{3^n} \qquad \text{b)} \quad \sum_{n=0}^{\infty} \frac{5^n}{3^n(n^4+2)} \qquad \text{c)} \quad \sum_{n=1}^{\infty} \left(\frac{1}{5} + \frac{1}{n} \right)^n \qquad \text{d)} \quad \sum_{n=0}^{\infty} \frac{1}{n^n}$$

(8P)

Problem 4: Prove that the field $\mathbb{C}$ together with the absolute value as norm is Cauchy complete. Recall: The norm of $z = (a, b) = a + ib$ is defined by $|z| := \overline{z} \cdot z = \sqrt{a^2 + b^2}$. (4P)