Math 3001 Analysis 1
Homework Set 3

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Course Instructor: Dr. Markus Pflaum

Contact Info: Office: Math 255, Telephone: 2-7717, e-mail: `markus.pflaum@colorado.edu`.

Problem 1: Show that the sequences $\left(\frac{n}{n+2}\right)_{n\in\mathbb{N}}$ and $\left(\frac{n}{2^n}\right)_{n\in\mathbb{N}}$ converge and determine their limits. (3P)

Problem 2: Prove that for every natural number $n \neq 0$:

a) $$\left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!} < 3,$$

b) $$\left(\frac{n}{3}\right)^n \leq \frac{1}{3}\, n!\,.$$

(6P)

Problem 3: Prove that for real $a, b \geq 0$ one has

$$\sqrt{a\,b} \leq \frac{1}{2}(a+b).$$

(2P)

Problem 4: Check which of the following sequences converges, and determine its limit, if it does:

a) $x_n = \frac{1+(-1)^n}{n}$ b) $x_n = \frac{n^3}{n^2+1}$

c) $x_n = \frac{n^2}{n^2+1}$ d) $x_n = \frac{n^2-n}{n^3+1}$

(4P)

Problem 5: Let $a, x_0 > 0$ be real numbers. Define the sequence $(x_n)_{n\in\mathbb{N}}$ recursively by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x\,n}\right) .$$

Show that the sequence $(x_n)_{n\in\mathbb{N}^*}$ is bounded and decreasing and that its limit is $\sqrt{a}$.
Hint: First show that $x_n > 0$ for all $n \in \mathbb{N}$. Then verify that $x_{n+1}^2 - a \geq 0$ for all $n \in \mathbb{N}$. After that you need to prove that the sequence $(x_n)_{n\in\mathbb{N}^*}$ is decreasing that is that $x_1 \geq x_2 \geq x_3 \geq \ldots$. Note that x_1 might be less, equal or greater than x_0, which means that monotony only holds from the index $n = 1$ on. Then put everything together and prove the claim. (5P)