Series

As $(a_n)_{n=1}^{\infty}$ is to $f(x)$

$\sum_{n=1}^{\infty} a_n$ is to $\int_a^{\infty} f(x)\,dx$

Def A partial sum $S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$

is a sum of the first N terms of a sequence.

Def A series $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n$ is the limit of partial sums

Ex $(a_n)_{n=1}^{\infty} = (1, 1, 1, \ldots)$.

$a_n = 1$

$S_1 = 1$

$S_2 = 1 + 1 = 2$

$S_3 = 1 + 1 + 1 = 3$

$S_N = N$

$\lim_{N\to\infty} S_N = \infty$ so the series $\sum_{n=1}^{\infty} a_n$ diverges to ∞.

Note that the sequence $(a_n)_{n=1}^{\infty}$ converges to 1

Ex $(a_n)_{n=1}^{\infty} = (1, 1, 0, 0, 0, \ldots)$

$a_n = 0$ for $n \geq 3$, $a_n = 1$ for $n = 1, 2$

$S_1 = 1$

$S_2 = 2$

$S_3 = S_4 = \ldots = 2$

$S_N = 2$ for sufficiently large N

so $\lim_{N\to\infty} S_N = 2$ and the Series $\sum_{n=1}^{\infty} a_n = 2$

Note that the sequence $(a_n)_{n=1}^{\infty}$ converges to 0.

Remark Never say "it converges"! what is "it"?

Ex $(a_n)_{n=1}^{\infty} = (1, -1, 1, -1, \ldots)$ Note the sequence $(a_n)_{n=1}^{\infty}$ diverges.

$a_n = (-1)^{n+1}$

$S_1 = 1$

$S_2 = 0$

$S_3 = 1$

$S_4 = 0$

$S_N = \begin{cases} 1 & N \text{ odd} \\ 0 & N \text{ even} \end{cases}$ so $\lim_{N\to\infty} S_N = \text{DNE}$ Hence the series $\sum_{n=1}^{\infty} a_n$ diverges.

Telescoping Series

Ex Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Note $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

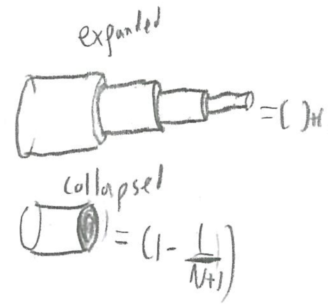
So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

So $s_1 = (1 - \frac{1}{2})$

$s_2 = (1 - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \frac{1}{3}) = 1 - \frac{1}{3}$

$s_3 = (1 - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}) + (\cancel{\frac{1}{3}} - \frac{1}{4}) = 1 - \frac{1}{4}$

$S_N = (1 - \cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}) + \dots + (\cancel{\frac{1}{N}} - \frac{1}{N+1}) = 1 - \frac{1}{N+1}$



So $\lim_{N \to \infty} S_N = 1$. Hence $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

In general

$$\sum_{n=1}^{\infty} b_n - b_{n+1} = \lim_{N \to \infty} b_1 - b_{N+1}$$

The harmonic Series

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

$S_1 = 1$

$S_2 = 1 + \frac{1}{2}$

$S_4 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) = 1 + \frac{2}{2}$

$S_8 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) = 1 + \frac{3}{2}$

$S_{2^n} > 1 + \frac{n}{2}$ So $S_N \to \infty$

Q: If $\sum_{n=1}^{\infty} a_n$ converges, what can we say about the sequence $(a_n)_{n=1}^{\infty}$

A: If $\sum_{n=1}^{\infty} a_n$ converges to S, the $S - S_N \to 0$ (the tail $\sum_{n=N+1}^{\infty} a_n$ goes to 0)

Note that $\lim_{n\to\infty} a_N = \lim_{N\to\infty} S - S_N = 0$ Hence $(a_n)_{n=1}^{\infty}$ converges to 0.

Therefore for a series $\sum_{n=1}^{\infty} a_n$ to converge, it is necessary for $a_n \to 0$,

However it is not a sufficient condition. Note $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So we have an (easy?) test for divergence,

Thm $\boxed{\text{The test for divergence}}$

If $\lim_{n\to\infty} a_n \neq 0$ or DNE, then $\sum_{n=1}^{\infty} a_n$ diverges.

Similar to integrals,

Thm Let $\sum a_n$ and $\sum b_n$ be convergent series, c a constant

Then (i) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Remark

These are almost aways false if $\sum a_n$ or $\sum b_n$ is divergent.