5.7 (pt. 1)

Trig Integrals

Sin/cos   Ex $\int \sin^5(x)\,dx = \int (1-\cos^2(x))^2 \sin(x)\,dx$

Let $u = \cos(x)$, $du = \sin(x)\,dx$

$\int \sin^5(x)\,dx = \int (1-u^2)^2\,du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$

$= \cos(x) - \frac{2}{3}\cos^3(x) + \frac{1}{5}\cos^5(x) + C$

Ex $\int_0^{\pi/2} \sin^2(x)\,dx = I$

Method 1: Integration by parts

Method 2: cos double angle formula $\left\{ \boxed{\begin{array}{l} \cos(2\theta) = \cos^2\theta - \sin^2\theta \\ \sin^2(\theta) = \frac{1-\cos(2\theta)}{2} \\ \cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \end{array}} \right.$

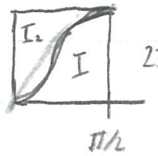
$I = \int_0^{\pi/2} \frac{1}{2}(1-\cos(2x))\,dx$

$= \frac{1}{2}\left[\int_0^{\pi/2} 1\,dx - \int_0^{\pi/2} \cos(2x)\,dx\right] = \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin(2x)\Big|_0^{\pi/2}\right] = \frac{\pi}{4}$

Method 3: Clever use of Geometry

By symmetry (or $u = \pi/2 - x$), $\underbrace{\int_0^{\pi/2} \sin^2(x)\,dx}_{I} = \underbrace{\int_0^{2} \cos^2(x)\,dx}_{I_2 = I}$

Note $\sin^2 + \cos^2 = 1$



$2I = I + I_2 = \int_0^{\pi/2} 1 = \pi/2$ So $I = \pi/4$

sec/tan: Note $\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$ or $\tan^2(\theta) + 1 = \sec^2(\theta)$

Ex $\int \tan^5(x)\sec^7(x)\,dx = \int (\sec^2(x)-1)^2 \sec^3(x)\,\sec(x)\tan(x)\,dx$

$= \int (u^2-1)^2 u^3\,du$ if $u = \sec(x)$, $du = \sec(x)\tan(x)\,dx$

Ex $\int \tan^4(x)\sec^4(x)\,dx = \int \tan^4(x)(1+\tan^2(x))^2 \sec^2(x)\,dx$

$= u^4(1+u^2)^2\,du$ if $u = \tan(x)$, $du = \sec^2(x)\,dx$

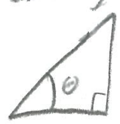
5.7 (pt. 2)

Trig substitution

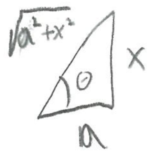
When to use: If you see $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$ in the integrand (i.e. $\int \frac{1}{\sqrt{x^2-1}}\,dx$) and no obvious u-sub.

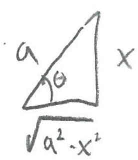
How to do:

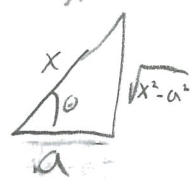
1) Draw a right triangle



2) Lable egdes so the one side lengths "work"





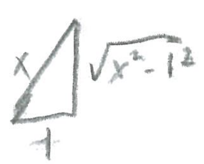


(or other variants)

$\left(\tan\theta = \frac{x}{a}\right)$ $\left(\sin(\theta) = \frac{x}{a}\right)$ $\left(\sec\theta = \frac{x}{a}\right)$

3) Do substitution

Ex $\int \frac{1}{\sqrt{x^2-1}}\,dx$



$\sec\theta = x$

$\sec\theta\tan\theta\,d\theta = dx$

$$\int \frac{1}{\sqrt{x^2-1}}\,dx = \int \cancel{\cot(\theta)}\sec\theta\,\cancel{\tan\theta}\,d\theta = \int \sec(\theta)\,d\theta$$

$$= \ln|\sec(\theta)+\tan\theta| + C \quad (\text{See Chapter 5.5})$$

$$= \ln\left|x+\sqrt{x^2-1}\right| + C$$

5.7 (pt. 3)

Partial Fractions

Question: How do we integrate functions like $\frac{x^2+x}{(x+1)^2(x^2+1)}$?

Answer: Separate into "nice" fractions

$$\frac{x^2+x}{(x+1)^2(x^2+1)} = \frac{A+B(x+1)}{(x+1)^2} + \frac{Cx+D}{x^2+1} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$$

Once we do this, we can integrate! (we'll solve for A, B, C, D late

$$\int \frac{x^2+x}{(x+1)^2(x^2+1)}\,dx = \underbrace{\int \frac{A}{(x+1)^2}\,dx}_{\text{Let } u_1 = x+1} + \underbrace{\int \frac{B}{x+1}\,dx}_{\text{Let } u_2 = x+1} + \int \frac{Cx}{x^2+1}\,dx + \int \frac{D}{x^2+1}\,dx$$

Let $u_3 = x^2+1$
$du_3 = 2x\,dx$

$$= A\int \frac{1}{u_1^2}\,du_1 + B\int \frac{1}{u_2}\,du_2 + \frac{1}{2}\int \frac{C}{u_3}\,du_3 + D\int \frac{1}{x^2+1}\,dx$$

[Recall $\int \frac{1}{x}\,dx = \ln|x|$ and $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$]

$$= -\frac{A}{x+1} + B\ln|x+1| + C\ln|x^2+1| + D\arctan(x)$$

solving for A, B, C, D.

1) Clear out the denominator

$$x^2+x = A(x^2+1) + B(x+1)(x^2+1) + (Cx+D)(x+1)^2$$

2) create a system of Equations and Solve

(Option 1) Plug in values of x:

$x=-1 \Rightarrow 1-1 = A(2) + B(0)(2) + (-C+D)(0)^2$ or $2A = 0$

$x=0 \Rightarrow 0 = A(1) + B(1)(1) + ((0)C+D)(1)^2$ or $A+B+D=0$

$x=1 \Rightarrow 1+1 = A(2) + B(2)(2) + (1C+D)(2)^2$ or $2A+4B+4C+4D=2$

$x=-2 \Rightarrow 4-2 = A(5) + B(-1)(5) + (-2C+D)(-1)^2$ or $5A-5B-2C+D=2$

(continued)

(why is $x=-1$ a nice number to choose?)

(Continued)

$$\begin{cases} 2A = 0 & (1) \\ A + B + D = 0 & (2) \\ 2A + 4B + 4C + 4D = 2 & (3) \\ 5A - 5B - 2C + D = 2 & (4) \end{cases}$$

By (1), $\boxed{A = 0}$

By (2) $0 + B + D = 0$ or $D = -B$

By (3) $0 + \cancel{4B} + 4C - \cancel{4B} = 2$ or $\boxed{C = \frac{1}{2}}$

By (4) $0 - 5B - \frac{2}{2} - B = 2$ or $-6B = 3$, $\boxed{B = -\frac{1}{2}}$

So $\boxed{D = \frac{1}{2}}$

(Option 2) Match powers of x

$$x^2 + x = Ax^2 + A + Bx^3 + Bx^2 + Bx + B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$

So $0x^3 = Bx^3 + Cx^3$

$x^2 = Ax^2 + Bx^2 + 2Cx^2 + Dx^2$

$x = Bx + Cx + 2Dx$

$0(1) = A + B + D$

or

$$\begin{cases} B + C = 0 & (1) \\ A + B + 2C + D = 1 & (2) \\ B + C + 2D = 1 & (3) \\ A + B + D = 0 & (4) \end{cases}$$

Remark Note that we can create the other system as

$$\begin{cases} -(1) + (2) - (3) \cancel{+(4)} \\ (4) \\ (1) + (2) + (3) + (4) \\ -8(1) + 4(2) - 2(3) + (4) \end{cases} = \begin{cases} 2A = 0 \\ A + B + D = 0 \\ 2A + 4B + 4C + 4D = 2 \\ 5A - 5B - 2C + D = 2 \end{cases}$$

So, both systems of equations are equivatent!

Solving

By (1), $C = -B$

By (3), $\cancel{B} - \cancel{B} + 2D = 1$ so $\boxed{D = \frac{1}{2}}$

By (2)-(4), $2C = 1$ so $\boxed{C = \frac{1}{2}}$ so $\boxed{B = -\frac{1}{2}}$

By (4), $A - \frac{1}{2} + \frac{1}{2} = 0$ so $\boxed{A = 0}$

Thus $\boxed{\int \frac{x^2 + x}{(x+1)^2(x^2+1)} dx = 0 - \frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x^2+1| + \frac{1}{2} \arctan(x)}$