5.10 # Indefinite Integrals

(Integrals with ∞)

Type I

Ex $\int_1^\infty \frac{1}{x^2}\,dx$

$$= \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b = \lim_{b\to\infty} -\frac{1}{b}+1 = 1$$

So $\boxed{\int_1^\infty \frac{1}{x^2}\,dx \text{ converges to } 1}$

In general $\int_1^\infty \frac{1}{x^p}\,dx \qquad p\neq 1$

$$= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx = \lim_{b\to\infty}\left(\frac{1}{p-1}\, x^{-p+1}\Big|_1^b\right) = \begin{cases}\infty & p<1\\ \frac{1}{p-1} & p>1\end{cases}$$

$\int_1^\infty \frac{1}{x}\,dx \qquad (p=1)$

$$= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|b| - \ln|1| = \infty$$

So $\int_1^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$ if $p>1$ and diverges to ∞ if $p\le$

Ex $\int_{-\infty}^0 xe^x\,dx = \lim_{a\to-\infty}\int_a^0 xe^x\,dx = \lim_{a\to-\infty}(xe^x)\Big|_a^0 - \int_a^0 e^x\,dx$

$$= \left[0 - \left(\lim_{a\to-\infty} ae^a\right)\right] - \left(e^0 - \cancel{\lim_{a\to-\infty} e^a}\right)$$

$$= -\lim_{x\to\infty}(-x)e^{-x} - 1$$

$$= \lim_{x\to\infty}\frac{x}{e^x} - 1$$

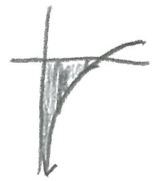
$$= \left(\lim_{x\to\infty}\frac{1}{e^x}\right) - 1 \qquad \text{(L'Hopital's Rul}$$

$$= -1$$

Indefinite Integrals (continued)

Type II

Ex $\int_0^1 \ln|x|\,dx$



$= \lim_{a\to 0^+} \int_a^1 \ln|x|\,dx$

$= \lim_{a\to 0^+} x\ln|x| - x \Big|_a^1$

$= 0 + \lim_{a\to 0^+} \left[a\ln(a) - (1-a) \right]$

$= -1 - \lim_{a\to 0^+} \frac{\ln(a)}{1/a}$

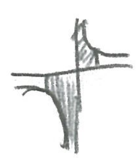
$= -1 - \lim_{a\to 0^+} \frac{1/a}{-1/a^2}$ (L'Hopital)

$= -1 - \lim_{a\to 0^+} -a$

$= -1$

So $\int_0^1 \ln|x|$ converges to -1

Ex $\int_{-1}^{1} \frac{1}{x}\,dx$



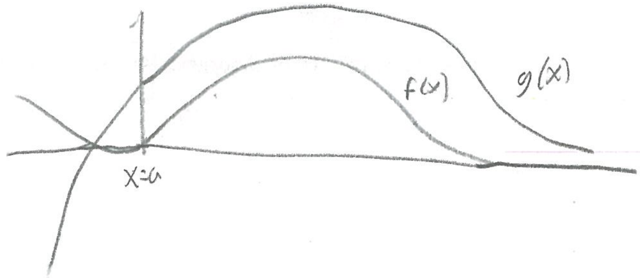
$= \int_{-1}^{0} \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$

$= \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x}\,dx + \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx = \infty - \infty = DNE$

Diverges

Indefinite Integrals (Comparison)

Thm Let $0 \le f(x) \le g(x)$ for $x \ge a$



Then

1) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges
2) If $\int_a^\infty f(x)\,dx$ diverges, the $\int_a^\infty g(x)\,dx$ diverges.

Thm Let $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$ and $0 < |L| < \infty$

strict inequalities

then either both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or both diverge

Remarks All we care about is "End Behavior"

Ex Does $\int_1^\infty \frac{2+\cos(x)}{\sqrt{x}}\,dx$ converge?

$0 \le \frac{1}{\sqrt{x}} \le \frac{2+\cos(x)}{\sqrt{x}}$ for $x \ge 1$

As $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges as a p-integral $(p = \frac{1}{2})$, $\int_1^\infty \frac{2+\cos(x)}{\sqrt{x}}\,dx$ diverges

Ex Does $\int_2^\infty \frac{1}{x^2-1}\,dx$ converge?

Note $\lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^2-1}} = 1$ and $0 < |1| < \infty$, and $\int_2^\infty \frac{1}{x^2}\,dx$ con

by limit comparison, $\int_2^\infty \frac{1}{x^2-1}\,dx$ converges.