Points and Vectors, Part II

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Recall, that if $\mathbb{R}^n = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n \text{ times}} = \{(x_1, \dots, x_n) \mid x_i \text{ is a real number}\}$, then the elements of $\mathbb{R}^n$ are denoted $\mathbf{x}$ or $\vec{x}$, and these stand for for $(x_1, \dots, x_n)$. We call them **points** or **vectors** depending on whether we consider them as positions or velocities/forces/etc. ('vector quantities', typically residing in phase space rather than configuration space, in physics lingo). There is a certain notation which is found in both physics and math books, that emphasises the *vector* part of the elements of $\mathbb{R}^n$, and it is the angle bracket notation:

$$\langle x_1, \dots, x_n \rangle \tag{0.1}$$

Let us, henceforth, denote *points* in $\mathbb{R}^n$ by bold or capital letters, and if we include components, let us use round brackets or parantheses:

$$\mathbf{x} = P = (x_1, \dots, x_n) \tag{0.2}$$

and let us, henceforth, denote *vectors* by the arrow notation and, in components, using angle brackets:

$$\vec{x} = \langle x_1, \dots, x_n \rangle \tag{0.3}$$

We need this in order to define the **displacement vector** from a point $P = (x_1, \dots, x_n)$ to another point $Q = (y_1, \dots, y_n)$ in $\mathbb{R}^n$, namely

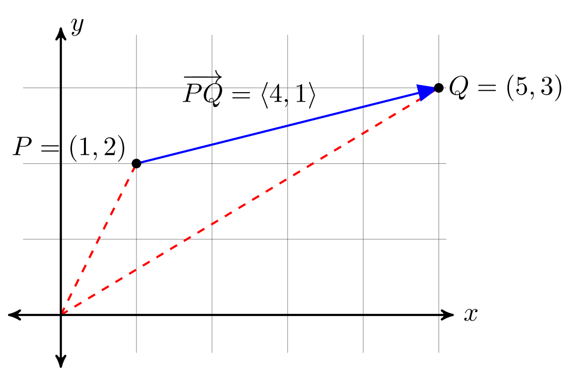
$$\begin{aligned} \overrightarrow{PQ} &= \vec{Q} - \vec{P} & (0.4) \\ &= \langle y_1, \dots, y_n \rangle - \langle x_1, \dots, x_n \rangle & (0.5) \\ &= \langle y_1 - x_1, \dots, y_n - x_n \rangle & (0.6) \end{aligned}$$

We *picture* the displacement vector $\overrightarrow{PQ}$ as emanating from the point P and ending in an arrow at the point Q.

Example 0.1 *Let us look at the case of* $\mathbb{R}^2$. *Suppose* $P = (1,2)$ *and* $Q = (5,3)$. *Then,*

$$\overrightarrow{PQ} = \vec{Q} - \vec{P} = \langle 5, 3 \rangle - \langle 1, 2 \rangle = \langle 5 - 1,\ 3 - 2 \rangle = \langle 4, 1 \rangle \tag{0.7}$$

and the picture is this:



Remark 0.2 *In fact, $\overrightarrow{PQ}$ should be pictured as emanating from the origin, but we* want *to think of $\overrightarrow{PQ}$ as emanating from P. We should, then, strictly speaking think of $\overrightarrow{PQ}$ as lying in a copy of $\mathbb{R}^n$ sitting above our position space $\mathbb{R}^n$ at the point P, that is we should think of $\overrightarrow{PQ}$ as lying in the set $\{P\} \times \mathbb{R}^n$, and thus*

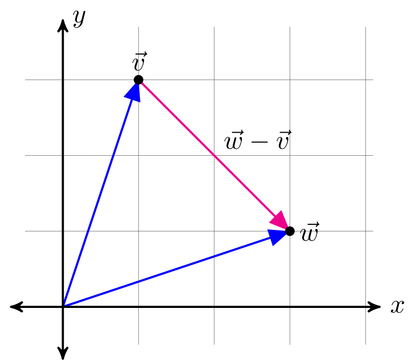
$$\overrightarrow{PQ} = (P, Q - P)$$

For example, if $P = (1,2)$ and $Q = (5,3)$, then

$$\overrightarrow{PQ} = \langle 4, 1\rangle = \big((1,2),\ (4,1)\big)$$

We will not nit-pick here, and we will simply conflate points and vectors in the strict sense, but we will picture *vectors as emanating from points in the underlying position space.* ■

Now suppose we are considering not two points P and Q, but two vectors $\vec{v}$ and $\vec{w}$. Then we can consider the **displacement vector** from $\vec{v}$ to $\vec{w}$. This is, in fact $\vec{w} - \vec{v}$, which is simply due to the fact that $\vec{w} = \vec{v} + (\vec{w} - \vec{v})$:



This of course harmonizes with our previous definition, $\overrightarrow{PQ} = \vec{Q} - \vec{P}$. We subtract our starting position vector from our ending position vector in both cases.

As usual, we define the **length** or **magnitude** of a vector $\vec{x}$ by (a generalization of) the Pythagorean theorem,

$$\|\vec{x}\| = \|\langle x_1, \dots, x_n\rangle\| = \sqrt{x_1^2 + \cdots + x_n^2} \tag{0.8}$$

It is clear that for any real number a we have

$$\|a\vec{x}\| = |a|\|\vec{x}\|$$

Example 0.3 *Let $P = (1,2)$ and $Q = (5,3)$ be points in the plane $\mathbb{R}^2$. Find the magnitude of the displacement vector from P to Q.*

Solution: Let $\vec{x} = \overrightarrow{PQ} = \vec{Q} - \vec{P} = \langle 4, 1\rangle$, then

$$\|\vec{x}\| = \|\langle 4, 1\rangle\| = \sqrt{4^2 + 1^2} = \sqrt{17}$$ ■

Lastly, let us use addition and scalar multiplicaton of vectors to decompose a given vector $\vec{x} = \langle x_1, \dots, x_n\rangle$ (or 'point' $\mathbf{x} = (x_1, \dots, x_n)$) into its **components** x_i:

$$\begin{aligned} \vec{x} &= \langle x_1, \dots, x_n\rangle \\ &= \langle x_1, 0, \dots, 0\rangle + \langle 0, x_2, 0, \dots, 0\rangle + \cdots + \langle 0, \dots, 0, x_n\rangle \\ &= x_1\langle 1, 0, \dots, 0\rangle + x_2\langle 0, 1, 0, \dots, 0\rangle + \cdots + x_n\langle 0, \dots, 0\rangle \end{aligned}$$

Let us define the **coordinate basis vectors**,

$$\begin{aligned}\mathbf{e}_1 &= \langle 1, 0, \ldots, 0\rangle \\ \mathbf{e}_2 &= \langle 0, 1, 0 \ldots, 0\rangle \\ &\vdots \\ \mathbf{e}_n &= \langle 0, \ldots, 0, 1\rangle\end{aligned}$$

Then, with this notation, we can more concisely write the above equality as

$$\vec{x} = \langle x_1, \ldots, x_n\rangle = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n \tag{0.9}$$

Example 0.4 *When $n = 2$, we have special notation, which is more commonly found in physics texts:*

$$\begin{aligned}\vec{i} &= \mathbf{i} = \mathbf{e}_1 = \langle 1, 0\rangle \\ \vec{j} &= \mathbf{j} = \mathbf{e}_2 = \langle 0, 1\rangle\end{aligned}$$

When $n = 3$, we also write:

$$\begin{aligned}\vec{i} &= \mathbf{i} = \mathbf{e}_1 = \langle 1, 0, 0\rangle \\ \vec{j} &= \mathbf{j} = \mathbf{e}_2 = \langle 0, 1, 0\rangle \\ \vec{k} &= \mathbf{k} = \mathbf{e}_n = \langle 0, 0, 1\rangle\end{aligned}$$

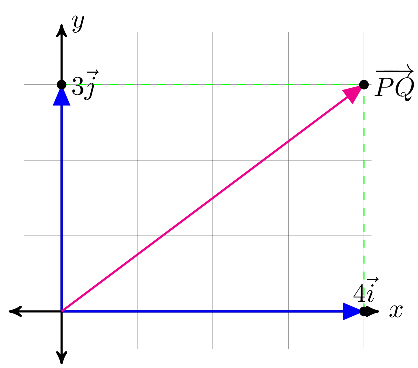
For example,

$$\langle 1, 3, -2\rangle = 1\vec{i} + 3\vec{j} - 2\vec{k} \tag{0.10}$$

is decomposed into its components. As another example,

$$\langle 4, 3\rangle = 4\vec{i} + 3\vec{j} \tag{0.11}$$

and this can be pictured as follows:



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Remark 0.5 *Frequently in undergraduate math and physics books vectors $\vec{x}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ are described entirely in terms of their components, so for example if $P = (2, 4, 10)$ and $Q = (3, 7, 6)$ are points in $\mathbb{R}^3$, the displacement vector $\overrightarrow{PQ}$ is written*

$$\overrightarrow{PQ} = (3-2)\vec{i} + (7-4)\vec{j} + (6-10)\vec{k} = 1\vec{i} + 3\vec{j} - 4\vec{k}$$ ■