Points and Vectors

Alex Nita

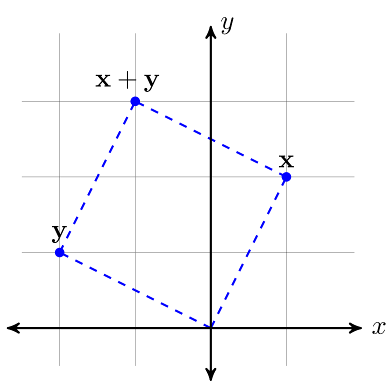
Let $\mathbb{R}^n = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n \text{ times}} = \{(x_1, \ldots, x_n) \mid x_i \text{ is a real number}\}$. We will sometimes use boldface notation for the n-tuples, $\mathbf{x}$ or 'vector' notation $\vec{x}$ for $(x_1, \ldots, x_n)$.

Example 0.1 *Our two main examples are the Euclidean plane* $\mathbb{R}^2$ *and Euclidean three-dimensional space* $\mathbb{R}^3$. *We usually denote* x_1 *by* x, x_2 *by* y, *and* x_3 *by* z, *so that* $\mathbf{x} = (x, y)$ *or* (x, y, z), *as the case may be.* ■

We define **addition** in $\mathbb{R}^n$ componentwise,

$$\mathbf{x} + \mathbf{y} = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \tag{0.1}$$
$$= (x_1 + y_1, \ldots, x_n + y_n) \tag{0.2}$$

Example 0.2 *Let us see what this means in* $\mathbb{R}^2$. *Take, say,* $\mathbf{x} = (1, 2)$ *and* $\mathbf{y} = (-2, 1)$. *Then* $\mathbf{x} + \mathbf{y} = (1 - 2, 2 + 1) = (-1, 3)$.



Thus we see that to reach $\mathbf{x} + \mathbf{y}$, *we may first go to* $\mathbf{x}$, *then go in the direction of* $\mathbf{y}$ *to get to* $\mathbf{x} + \mathbf{y}$, *or else we may to to* $\mathbf{y}$ *first and then go in the direction of* $\mathbf{x}$. *This shows the algebraically obvious fact that* $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. ■

In general, we have **commutativity of addition**:

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) \\ &= (y_1 + x_1, \ldots, y_n + x_n) \\ &= \mathbf{y} + \mathbf{x} \end{aligned} \tag{0.3}$$

which, as we saw in the previous example, geometrically means that we may 'get to' $\mathbf{x}+\mathbf{y}$ in any order we like, first along $\mathbf{x}$ then along $\mathbf{y}$, or along $\mathbf{y}$ first and then along $\mathbf{x}$.

Another obvious fact about our definition of addition in $\mathbb{R}^n$ is that it is **associative**:

$$(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z}) \tag{0.4}$$

which again follows from the same associativity holding in each component, $(x_i + y_i) + z_i = x_i + (y_i + z_i)$.

It is also clear that **zero**, the element $\mathbf{0} = (0, \ldots, 0)$, satisfies

$$\mathbf{0}+\mathbf{x} = \mathbf{x}+\mathbf{0} = \mathbf{x} \tag{0.5}$$

for all $\mathbf{x}$ in $\mathbb{R}^n$. Moreover, the **negative** element of $\mathbf{x}$ in $\mathbb{R}^n$, defined by

$$-\mathbf{x} = (-x_1, \ldots, -x_n) \tag{0.6}$$

satisfies

$$(-\mathbf{x})+\mathbf{x} = \mathbf{x}+(-\mathbf{x}) = \mathbf{0}$$

which we may more compactly write $-\mathbf{x}+\mathbf{x} = \mathbf{x}-\mathbf{x} = \mathbf{0}$. That is, we may define **subtraction** of elements of $\mathbb{R}^n$ by addition of negatives:

$$\mathbf{x}-\mathbf{y} = \mathbf{x}+(-\mathbf{y}) \tag{0.7}$$

Let us now define **scalar multiplication** of elements of $\mathbb{R}^n$. That means, we will define multiplication of $\mathbf{x}$ by a real number a. As with addition, we define this componentwise:

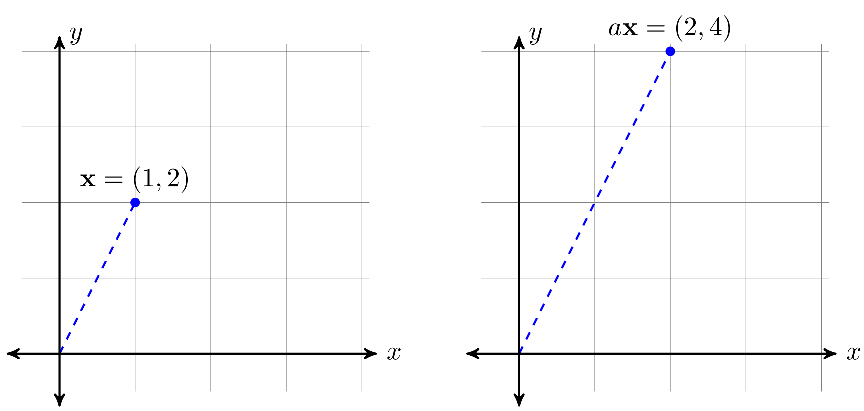
$$a\mathbf{x} = a(x_1, \ldots, x_n) = (ax_1, \ldots, ax_n) \tag{0.8}$$

It is clear that $1\mathbf{x} = \mathbf{x}$ and $0\mathbf{x} = \mathbf{0}$. Moreover, we have **associativity of scalar multiplication**, $a(b\mathbf{x}) = (ab)\mathbf{x}$, and we have **distributivity** of scalar multiplication over addition:

$$\begin{aligned} a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x}+a\mathbf{y} \\ (a+b)\mathbf{x} &= a\mathbf{x}+b\mathbf{x} \end{aligned}$$

for all real numbers a and b and all elements $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$.

Example 0.3 *Let us see what this means in $\mathbb{R}^2$. Take, say, $\mathbf{x} = (1, 2)$ and $a = 2$. Then $a\mathbf{x} = 2(1, 2) = (2 \cdot 1,\ 2 \cdot 2) = (2, 4)$.*



Thus geometrically scalar multiplication has the effect of scaling the distance of the element $\mathbf{x}$ from the origin $\mathbf{0}$. ■

Let us now touch upon the distinction between points and vectors. We will call elements of $\mathbb{R}^n$ **points** when we think of them as positions in n-space, and we will call them **vectors** when we think of them as having direction (in which case we will put an arrow in illustrations). The **length** or **magnitude** of a point/vector $\mathbf{x}$ in $\mathbb{R}^n$ will be defined as the distance of $\mathbf{x}$ from the origin $\mathbf{0}$, and will be denoted by x or $\|\mathbf{x}\|$,

$$x \quad \text{or} \quad \|\mathbf{x}\| = d(\mathbf{x}, \mathbf{0}) = \sqrt{x_1^2 + \cdots + x_n^2} \tag{0.9}$$

Remark 0.4 *We remark here that the definitions of addition and scalar multiplication in $\mathbb{R}^n$ make $\mathbb{R}^n$ into a* ***vector space****. If we think of the vector space $\mathbb{R}^n$ as a position space for particles, then we tend to think of its elements as points, which happen to also be vectors, whereas if we think of the vector space $\mathbb{R}^n$ as phase space or something analogous, where velocities of particles and forces act, then we think of $\mathbb{R}^n$ as consisting* entirely *of vectors. The idea of* translation invariance *of vectors in $\mathbb{R}^n$ is an amalgam of these two notions. In such a situation we think of $\mathbb{R}^n$ as having both points and vectors in it, with the vectors moving around but staying unchanged in magnitude and direction. This actually means that at each 'point' $\mathbf{x}$ we attach a 'vector' $\mathbf{v}$ so that the vector 'emanates' from $\mathbf{x}$. But this really means that at $\mathbf{x}$ we have attached a* copy *of $\mathbb{R}^n$, whose elements we treat as 'vectors', and we superimpose this copy onto our 'position space'. The fact that there is such a copy at each point $\mathbf{x}$ of our position space means that we can think of the vector $\mathbf{v}$ as 'moving' from point to point, but really it's just another copy of $\mathbf{v}$. This, however, is a technicality, and I only include it to clarify the math going on here.* ■