Examples of Poisson Random Variables

(1) # misprints on a page / in a section of a book ($n = $ # letters per page, p small)

(2) # of people in a given community to survive age 100

(3) # of wrong telephone numbers dialed per day

(4) # customers entering a post office per day

(5) # vacancies in a year in federal judicial system

(6) # α-particles emitted per unit time interval by a radioactive substance

These are all approximately Poisson, because the Poisson pmf approximates the binomial pmf.

ex. 1 (typographical errors) If $\lambda = \frac{1}{2}$, what is the probability of at least one error on a page?

A: $X = $ # errors per page (on a given page) $\Longrightarrow P(X \geq 1) = 1 - P(X=0)$

$$\approx 1 - e^{-\lambda} \cdot \frac{\lambda^0}{0!} = \boxed{1 - e^{-\lambda}}$$

ex. 2 (defective item manufactured in plant)

Suppose the probability of a certain item produced by a machine in a factory is

$$p = 0.1$$

What is the probability that a 10-item sample contains at most 1 defective item?

A: $P(X \le 1) = \overbrace{P(X=0) + P(X=1)}^{\text{binomial}}$

$$= p(0) + p(1)$$

$$= \binom{10}{0} p^0 (1-p)^{10} + \binom{10}{1} p^1 (1-p)^9$$

$$= \left(\frac{9}{10}\right)^{10} + \cancel{10} \cdot \frac{1}{\cancel{10}} \cdot \left(\frac{9}{10}\right)^9$$

$$\boxed{\begin{aligned} &= \left(\frac{9}{10}\right)^9 \left(\frac{9}{10} + 1\right) \\ &\approx 0.7361 \end{aligned}}$$

& Poisson approx.: $P(X \le 1) = P(X=0) + P(X=1)$

$$\approx e^{-\lambda} \cdot \frac{\lambda^0}{0!} + e^{-\lambda} \cdot \frac{\lambda^1}{1!}$$

$\left(* \ \lambda = np = 10 \cdot \frac{1}{10} = 1 \right)$

$$= e^{-\lambda}(1+\lambda) = \boxed{2e^{-2} \approx 0.7358}$$

ex. 3 (α-particle emission)

If we know from experiments that on average 3.2 α-particles are emitted in 1 sec. by 1 gram of radioactive material, what is a good approx. that no more than 2 α-particles will be emitted?

A: 1 gram means large n (# of atoms), each of which has prob. $\frac{3.2}{n}$ of being emitted as α-particles, so $\lambda = np = \cancel{n} \cdot \frac{3.2}{\cancel{n}} = 3.2$.

Therefore,

$$P(X \leq 2) \approx \sum_{i=0}^{2} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} + e^{-\lambda} \cdot \lambda + e^{-\lambda} \cdot \frac{\lambda^2}{2}$$

$$= e^{-3.2}\left(1 + 3.2 + \frac{(3.2)^2}{2}\right)$$

$$\approx \boxed{0.3799}$$

ex Flip a biased coin n times, p = prob. H.
What is the probability of k consecutive heads?

A: For each $i = 1, \ldots, n-k+1$ let

H_i = event that flips i through $i+k-1$ are all H

Then, $P(k \text{ consecutive } H) = P(\text{at least one } H_i)$.
Now, clearly $P(H_i) = p^k$, which should be small. But these events are not indep., not even weakly, because e.g.

$$P(H_2 | H_1) = \frac{P(H_1 \cap H_2)}{P(H_1)} = \frac{p^{k+1}}{p} = p$$

which is not small enough for independence (even weak indep.).

Let E_i = event that flips i through $i+k-1$ are all H and flip $i+k$ is T $\Big\}$ $i=1,\ldots,n-k$

& E_{n-k+1} = flips $n-k+1$ through n are all H

and observe that

$$P(E_i) = p^k(1-p), \quad i=1,\ldots,n-k$$

$$P(E_{n-k+1}) = p^k$$

When p^k is small, $P(E_i)$ is small, &

$$i \neq j \implies P(E_i|E_j) = \begin{cases} P(E_i), & \text{if } E_i \,\&\, E_j \text{ have nonoverlapping sequences} \\ 0, & \text{if non-overlapping} \end{cases}$$

(ex. $k=4$, $n=10$: E_1 = THHHHTTHHT
E_2 = HHHHHTTTTT overlap

$\implies P(E_2|E_1) = P(E_1)$)

$\implies$ close to unconditional

Letting $N = \#$ of events E_i, our random variable, we have

$$E[N] = \sum_{i=1}^{n-k+1} i\,p(i)$$

$$= \sum_{i=1}^{n-k+1} i\,P(N=i)$$

$$= \sum_{i=1}^{n-k+1} P(E_i)$$

$$= (n-k)p^k(1-p) + p^k = \lambda$$

$$\Longrightarrow \quad P(N=0) \approx e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-\lambda}$$

$$= e^{-(n-k)p^k(1-p)+p^k}$$

Now consider the ~~[illegible]~~ random variable

$L_n =$ largest # consecutive heads

Then, $P(L_n < k) = P(N=0) \approx e^{-(n-k)p^k(1-p)+p^k}$