IX

EUCLID

We have little or no detailed information about the life and personality of any of the great mathematicians of Greece. Euclid himself is no exception. Practically all that we know of him comes from Proclus' summary, but it is evident that Proclus based himself upon inference more than anything else. He speaks of him as 'Euclid who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors'; and he infers that he lived in the time of the first Ptolemy (306–283 B.C.) because Archimedes mentions him, and Archimedes came immediately after the first Ptolemy; Proclus evidently knew nothing of his birthplace or of the dates of his birth and death. He only adds the story of Euclid's reply to Ptolemy that there is no royal road to geometry.

The Arabs, who eagerly assimilated Greek geometry, so far adopted Euclid that they wished to connect him with the East (the same predilection made them describe Pythagoras as a pupil of the wise Salomo, Hipparchus as the exponent of Chaldaean philosophy or as 'the Chaldaean', Archimedes as an Egyptian, and so on); according to the Arabs, Euclid was the son of Naucrates and grandson of Zenarchus, a Greek born at Tyre and domiciled at Damascus. They identified him so completely with geometry that they interpreted his name (which they pronounced variously as Uclides or Icludes) as meaning the

'key of geometry' (from *Ucli*, a key, and *Dis*, a measure or, as some say, geometry); and again they said that the Greek philosophers used to put up on the doors of their schools the well-known notice 'Let no one come to our school who has not first learnt the Elements of Euclid' (an obvious adaptation of Plato's ἀγεωμέτρητος μηδεὶς εἰσίτω, 'let no one enter who has no geometry').

Euclid probably received his mathematical training from the pupils of Plato in Athens; but he himself founded a school at Alexandria, where, according to Pappus, Apollonius of Perga afterwards 'spent a long time with the pupils of Euclid'.

There is another story of him which one would like to believe true. According to Stobaeus, some one who had begun to read geometry with Euclid had no sooner learnt the first theorem than he asked, 'What shall I get by learning these things?' whereupon Euclid called the slave and said, 'Give him threepence, since he must needs make gain out of what he learns.'

Pappus praises Euclid for his modesty and his consideration for others. Thus Euclid, he says, regarded Aristaeus as deserving credit for the discoveries he had made in conics (Aristaeus had written a treatise on *Solid Loci*, that is to say, on conics, no doubt regarded as loci), and made no attempt to anticipate him or to construct afresh the same system, such was his scrupulous fairness and his exemplary kindliness to all who could advance mathematical science to however small an extent. Pappus is here comparing Euclid with Apollonius, to the disadvantage of Apollonius, whom he evidently regarded as too self-assertive, probably because in the prefaces to the Books of his *Conics* Apollonius seemed to Pappus to give too little credit to Euclid for his earlier work on the same subject.

But we can believe Pappus' testimony to Euclid's character, for his extant works betray no sign of any claim to be original. In the *Elements*, for instance, there is not a word of preamble; he plunges at once into his subject: 'A point is that which has no part'! And although he made great changes in the exposition, altering the arrangement of whole Books, redistributing propositions between them, and inventing new proofs where the new order made the earlier proofs inapplicable, it is safe to say that Euclid made no more alterations than his own acumen and the latest special researches (such as Eudoxus' theory of proportion) showed to be necessary to make the treatment more scientific. He even showed unnecessary respect for tradition, as when he retained certain definitions never afterwards used, and when in solitary passages in Book III he permits himself to speak of the 'angle *of* a semicircle' and the 'angle *of* a segment'.

THE *ELEMENTS*

Euclid has always been known almost exclusively as the author of the *Elements*, ὁ στοιχειωτής, as the Greeks from Archimedes onwards called him instead of using his name. This wonderful book, notwithstanding its imperfections, remains the greatest elementary text-book in mathematics that the world is privileged to possess. Scarcely any other book except the Bible can have circulated more widely the world over or been more edited and studied. Immediately on its appearance it superseded all other Elements, and that so completely that no others have survived. Archimedes already cites propositions by the Book and number; so do all later Greek mathematicians. Even in Greek times the most accomplished mathematicians, e.g. Heron and Pappus (to say

nothing of Porphyry, Simplicius, and Proclus) wrote commentaries. The great Apollonius of Perga was moved by Euclid's work to discuss the first principles of geometry. His *General Treatise* (ἡ καθόλου πραγματεία), as it was called, seems to have contained suggestions for improvement, e.g. a new definition of an angle and alternative constructions for the problems of I. 10, 11, 23. These last seem to show that Apollonius wished to give a more *practical* turn to the beginnings of the subject, herein anticipating the tendency which has of late been so much in evidence. So true is it that 'there is nothing new under the sun' or, as Aristotle forcibly puts it more than once, 'it is not once nor twice but times without number that the same ideas make their appearance in the world'. Apollonius' attempt to prove the Axioms (if we may judge by what Proclus gives as his attempted proof of Ax. 1) was thoroughly misconceived.

Even in ancient times certain things in Euclid were the subject of acute controversy. We know from Aristotle that in his time the theory of parallels had not yet been put on a scientific basis. Euclid seems to have been the first to see the necessity of some postulate and to formulate one. But the fifth Postulate was found a great stumbling-block. Why should a fact which is the converse of an ordinary proposition proved by Euclid himself not be proved? Some tried to prove it, like Ptolemy, Proclus, and (according to Simplicius) one Diodorus, as well as 'Aganis'; we have three of these attempted proofs, which of course all, tacitly or otherwise, make some equivalent assumption. Posidonius and Geminus substituted an *equidistance* theory of parallels.

Zeno of Sidon objected even to Eucl. I. 1 as not being conclusive unless we assume that neither two straight lines nor two circumferences can have a common part, and Posidonius wrote a whole book to controvert Zeno. It was the habit of the Epicureans, says Proclus, to ridicule Eucl. I. 20 (proving that two sides of a triangle are together greater than the third) as being evident even to an ass and requiring no proof. But the Epicureans and Sceptics objected to the whole of mathematics, mainly on the ground of the opposition between the fundamental hypotheses of mathematics and the data of sense. There are no such things, they said, as mathematical points, lines, &c. Even if points exist, you cannot make up a line out of points. It is absurd to define a line as that which, if it be turned about one of its extremities, will always touch a plane; a line being, say, length without breadth, and therefore an unsubstantial thing, cannot be turned round at all; and so on. I mention these things in order to show that mathematicians had to contend with Philistines and others even in Greece.

Cicero is the first Latin author to mention Euclid; but it is not likely that in his time Euclid had been translated into Latin. As Cicero says elsewhere, the Romans did not care for geometry except so far as it was useful for measurements and calculations. Philosophers studied Euclid, but no doubt in the original Greek. Martianus Capella speaks of the effect of the mention of the problem 'how to construct an equilateral triangle on a given straight line' among a company of philosophers, who, recognizing the first proposition of the *Elements*, straightway break out into encomiums on Euclid. Beyond a fragment in a Verona palimpsest of a free reproduction of some propositions from Books XII and XIII dating apparently from the fourth century, there is no trace of any Latin version before Boëtius (born about A.D. 480), to whom Magnus

Aurelius Cassiodorus and Theodoric attribute a translation of Euclid. The so-called geometry of Boëtius (which is, however, only a compilation from various sources, put together in the eleventh century) is anything but such a translation, for it contains only the definitions of Book I, the five Postulates, three Axioms, certain definitions from Books II, III, IV, the enunciations (without proofs) of Book I, ten propositions of Book II and a few from Books III, IV, and, lastly, a passage indicating that the editor will now give something of his own, which proves to be a literal translation of the proofs of Eucl. I. 1–3. This shows that the pseudo-Boëtius had a Latin translation of Euclid from which he extracted these proofs. Moreover, the text of the definitions from Book I shows traces of correct readings which are not found even in the Greek MSS. of the tenth century, but which appear in Proclus and other ancient sources.

All the Greek texts of the *Elements* before Peyrard's (1814–18) were based on MSS. containing Theon's recension (fourth century A.D.). They purport in their titles to be 'from the edition of Theon' or 'from the lectures of Theon', and they contain, in the proposition VI. 33, an application to *sectors* of circles which Theon in his commentary on Ptolemy's *Syntaxis* claims as having been proved by himself in his edition of the *Elements*. When, therefore, Peyrard found in the Vatican the great MS. gr. 190 (now known as P) which contained neither the words from the titles of other MSS. quoted above nor the addition to VI. 33, it was clear that here was what, on the face of it, represented a more ancient edition than Theon's. This is confirmed by the fact that the copyist of P (or rather its archetype) had the two recensions before him and systematically gave the preference to the earlier one; thus the first hand in P has on XIII. 6 a marginal note that 'this theorem is not given in most copies of the new edition, but is found in those of the old'. The *editio princeps* of the Greek text by Simon Grynaeus (Basel 1533) was based on two manuscripts of the sixteenth century which are among the worst. Gregory in his great edition (Oxford 1703) followed the *editio princeps* in the main, only consulting the manuscripts bequeathed by Savile to the University where the Basel text differed from the Latin translation by Commandinus. Even Peyrard only corrected the Basel text by means of P, instead of rejecting it altogether and starting afresh. E. F. August's edition (1826–9) followed P more closely and used the Viennese MS. gr. 103 as well. It was reserved for Heiberg to bring out the new and authoritative text based on P and the best of the Theonine MSS., and taking account of external sources such as Heron and Proclus. Authors earlier than Theon, e.g. Heron, generally agree with our best manuscripts, and Heiberg concludes that the *Elements* were most spoiled by interpolations about the third century, since Sextus Empiricus (about A.D. 200) had a correct text, while Iamblichus had an interpolated one.

A remarkable difference between the best and the inferior manuscripts is in the number and arrangement of the Postulates and Axioms. Our ordinary editions based on Simson had three Postulates and twelve Axioms. Of the twelve Axioms, the eleventh (that all right angles are equal) is, in the genuine text, the Fourth Postulate, and the twelfth is the Fifth Postulate (the Parallel-Postulate) The genuine Postulates are thus five in number. Of the Axioms or Common Notions Heron recognized only three, and Proclus only these and two others (that things which coincide are equal, and that the whole is greater than the part); all the rest, therefore, are probably interpolated, including the assumption that 'two straight lines cannot enclose a space'.

The first Latin translations which we possess in a complete form were made not from the Greek but from the Arabic. The Caliphs al-Manṣūr (754–75) and al-Ma'mūn (813–33) obtained from the Byzantines manuscripts of Euclid among other authors. The *Elements* were translated in the reign of ar-Rashīd (786–809) by al-Ḥajjāj b. Yūsuf b. Maṭar, who also made a second version abridged from the other, but with corrections and explanations, for al-Ma'mūn; six Books of the latter version survive in the Codex Leidensis 399. 1, which has been edited (as to four Books) with Latin translation by Besthorn and Heiberg. The next translation was by Isḥāq b. Ḥunain b. Isḥāq al-'Ibādī (died 910); this translation, as revised by Thābit b. Qurra (died 901), exists in two manuscripts in the Bodleian. Isḥāq's version seems to be a model of good translation; while attempting to get rid of difficulties and unevennesses in the Greek text, the translator gave a faithful reproduction of it. The third extant Arabic version, by Naṣīraddīn aṭ-Ṭūsī (born at Ṭūs in Khurāsān in 1201), is not a translation but a rewritten version based on the older Arabic translations.

The known Latin translations begin with that of Athelhard, an Englishman, of Bath, made about A.D. 1120. It was made from the Arabic, as is proved by the occurrence in it of Arabic words, but Athelhard must also have had before him a translation of (at least) the enunciations based ultimately on the Greek text, and going back to the old Latin version used by the Pseudo-Boëtius and the *Gromatici*. But some sort of translation, or fragments of one, must have reached England earlier still, namely about 924–30, if we may judge by the old English verses:

The clerk Euclide on this wyse hit fonde
Thys craft of gemetry yn Egypte londe
Yn Egypte he tawghte hyt ful wyde,
In dyvers londe on every syde.
Mony erys afterwarde y understonde
Yer that the craft com ynto thys londe.
Thys craft com into England, as y yow say,
Yn tyme of good Kyng Adelstone's day.

Next Gherard of Cremona (1114–87) is said to have translated the '15 Books of Euclid' from the Arabic, as he undoubtedly translated an-Nairīzī's Commentary on Books I–X. This translation of the *Elements* was formerly supposed to be lost, but Björnbo claims (1904) to have discovered it in manuscripts at Paris, Boulogne-sur-Mer, Bruges, and (as regards Books X–'XV') at Rome. Gherard's translation was independent of Athelhard's, and gives a word for word rendering of an Arabic manuscript containing a revised and critical edition of Thābit's version.

The third translation from the Arabic was made about 150 years after that of Athelhard by Johannes Campanus. It was not independent of Athelhard's, as is clear from the fact that, in all manuscripts and editions, the definitions, postulates, and axioms and the 364 enunciations are word for word identical in Athelhard and Campanus. But Campanus' translation is the clearer and more complete of the two; the arrangement is also different in that Athelhard regularly puts the proofs before the enunciations, instead of following the usual order. It may be that Campanus used Athelhard's translation, but altered and improved it by means of other Arabic originals.

Gherard of Cremona, in addition to translating the

Elements and an-Nairīzī's commentary thereon, made a whole series of translations from the Arabic of Greek treatises, including the *Data* of Euclid, the *Sphaerica* of Menelaus and Theodosius, and the *Syntaxis* of Ptolemy; he also translated Arabian geometrical works such as the *Liber trium fratrum* and, in addition, the algebra of Muḥammad b. Mūsā. The interest in Greek and Arabian mathematicians thus aroused quickly led to fruitful results in the brilliant works of Leonardo of Pisa (Fibonacci). Leonardo first published in 1202, and then later (1228) brought out an improved edition of, his *Liber Abaci* giving the whole of arithmetic and algebra as known to the Arabs, but in a free and independent style of his own. In like manner, in his *Practica geometriae* (1220), he collected (1) all that the *Elements* of Euclid and Archimedes' works *On the Measurement of a Circle* and *On the Sphere and Cylinder* had taught him of the measurement of plane figures bounded by straight lines, solids bounded by planes, the circle, and the sphere respectively, (2) divisions of figures after the manner of Euclid's book *On Divisions* (of figures), but carried further, (3) some trigonometry. Leonardo is, however, a solitary figure in a waste which extended over the next three centuries; it is as if the talent he had left to the Latin world had lain hidden in a napkin and earned no interest. Roger Bacon (1214–94), though no doubt he exaggerated a little, is witness to the neglect of geometry in education. The philosophers of his day, he says, despised geometry, languages, &c., declaring that they were useless; and people in general, finding no utility in any science such as geometry, could hardly (unless they were boys forced to it by the rod) be induced to study so much as three or four propositions of Euclid, while the fifth proposition was called *Elefuga* or *fuga* *miserorum*, a punning identification of 'escape from the Elements' with 'escape from troubling' (ἔλεος).

In the Universities in this country and abroad during the fourteenth and fifteenth centuries little geometry was required from candidates for degrees. To have attended lectures on a few books (not more than six) of the *Elements* was a usual qualification. At Oxford in the middle of the fifteenth century two Books of Euclid were read, and no doubt the Cambridge course was similar.

With the issue, however, of the first printed editions of the *Elements* the study of Euclid received a great impetus.

The first printed edition was published at Venice by Erhard Ratdolt in 1482. It contained Campanus' translation from the Arabic already mentioned. This beautiful and very rare book was not only the first printed edition of Euclid, but also the first printed mathematical book of any importance. In the margins of $2\frac{1}{2}$ inches were printed the figures of the propositions. Ratdolt says in his dedication that at that time, although books by ancient and modern authors were being printed every day in Venice, little or nothing mathematical had appeared; this fact he puts down to the difficulty caused by the diagrams, which no one up to that time had succeeded in printing; he adds that after much labour he had discovered a method by which figures could be produced as easily as letters. How eagerly the opportunity of spreading geometrical knowledge was seized is proved by the number of editions which followed in the next few years. Even 1482 saw two forms of the book, though they only differ in the first sheet. Another edition appeared at Ulm in 1486, and another at Vicenza in 1491.

Bartolomeo Zamberti (Zambertus) was the first to bring out a translation of the whole of the *Elements* from the Greek; this appeared at Venice in 1505. The most important Latin translation is, however, that of Commandinus (1509–75), who followed the Greek text more closely than his predecessors and added some ancient scholia, as well as good notes of his own; this translation, which appeared in 1572, was the basis of most translations down to that of Peyrard, including Simson's, and therefore of all those editions, numerous in England, which gave Euclid 'chiefly after the text of Dr. Simson'.

The first complete English translation (1570) is that of Henry Billingsley, who was Sheriff of London in 1584 and was elected Lord Mayor, on a death vacancy, on December 31, 1596. This is a monumental work of 928 folio pages, with a preface by John Dee and notes extracted from all the most important commentaries from Proclus down to Dee himself, a magnificent tribute to Euclid. About the same time Henry Savile began to give unpaid lectures on the Greek geometers; those on Euclid (of 1620) do not extend beyond I. 8, but they are valuable because they grapple with the difficulties connected with the preliminary matter, the definitions, &c., and the tacit assumptions made in the first propositions. It was, however, in the period from about 1660 to 1730, during which Wallis and Halley were Professors at Oxford, and Barrow and Newton at Cambridge, that the study of Greek mathematics was at its height in England. Barrow's admiration for Euclid was unbounded. His Latin version (*Euclidis Elementorum Libri XV breviter demonstrati*) appeared in 1655, and several more editions followed down to 1732; the first English edition appeared in 1660 and was followed by others in 1705, 1722, 1732, and 1751. We are told that Newton, when he first bought a Euclid in 1662 or 1663, thought it a 'trifling book', as the propositions seemed to him obvious; afterwards, however, on Barrow's advice, he studied the *Elements* carefully and derived, as he himself stated, great benefit therefrom. The unique status of Euclid as a text-book in England, which lasted until recently, may perhaps be said to date from the publication of Robert Simson's *Elements of Euclid*, which first appeared both in Latin and in English in 1756. It was a full translation, mainly based on Commandinus, of Books I–VI, XI, and XII, but enriched by valuable notes and suggestions for improvements; it was the basis of most editions down to Todhunter's.

As Euclid's propositions are in the form which is recognized as classical, though that form was not invented by him, it will be useful to give here some account of certain technical terms used by the Greeks in connexion with their form of exposition.

In its completest form a proposition contained six parts: (1) the πρότασις, or *enunciation*, in general terms; (2) the ἔκθεσις or *setting-out*, which states the data, e.g. the particular lines or figures given and denoted by letters, on which the argument is to be illustrated; (3) the διορισμός, *definition* or *specification*, which is a restatement of what it is required to do or prove, but in terms of the particular data, the object being to fix our ideas; (4) the κατασκευή, *construction* or *machinery*, which includes any additions to the original figure by way of construction in order to facilitate the proof; (5) the ἀπόδειξις, the *proof* itself; (6) the συμπέρασμα, or *conclusion*, which reverts to the enunciation and states (often in the same general terms) what has been proved or done. A particular proposition may be without some of these parts (e.g. no *construction* beyond the data may be necessary), but three parts are indispensable to all propositions, the enunciation, proof, and conclusion.

A special term was used in connexion with problems, namely διορισμός, the same word as we have seen above in (3). There it means a restatement, in terms of particular data, of what it is required to prove or do; here it means the statement of the conditions under which the solution of the problem is possible, or, in its most complete form, a criterion as to 'whether what is sought is impossible or possible, and how far it is practicable and in how many ways' (Proclus). It is introduced, like the other kind of διορισμός, by the phrase δεῖ δή, 'it is thus necessary' (or 'required'). Cf. I. 22, 'Out of three straight lines which are equal to three given straight lines to construct a triangle: thus it is necessary that two of the given straight lines taken together in any manner should be greater than the remaining straight line'.

The *Elements* is a *synthetic* treatise, proceeding straight forward from the known and simple to the unknown and more complex; *analysis*, therefore, which reduces the unknown, or less known, and more complex to the known, has no place in the exposition, though no doubt it played its part in the discovery of the proofs. But *reductio ad absurdum*, a method of proof to which Euclid often has to resort, is a variety of analysis; for analysis begins with the *reduction* (ἀπαγωγή) of the proposition required to be proved, which we hypothetically assume to be true, to something simpler which we can immediately recognize as true or false; the case where the reduction leads to a conclusion obviously false is the *reductio ad absurdum* (ἡ εἰς τὸ ἀδύνατον ἀπαγωγή).

What we call a *corollary* was for the Greeks a *porism* (πόρισμα), something provided or ready-made, i.e. some result incidentally revealed in the course of the demonstration of the main proposition. The name *porism* was also applied to a special kind of substantive proposition, as in

Euclid's three Books of *Porisms*, now lost (see pp. 262–5 below).

The word *lemma* means simply something *assumed*. Archimedes uses it of what is now known as the Axiom of Archimedes, the principle used by Eudoxus and others in the method of exhaustion. More commonly it is an auxiliary proposition requiring proof, but assumed in the place where it is wanted in the main proof for convenience' sake only, in order that the argument may not be interrupted or unduly lengthened. It may be proved in advance, but is often left over to be proved afterwards (ὡς ἑξῆς δειχθήσεται, 'as will be proved immediately').

The content of the 'Elements'.

Book I of the *Elements* necessarily begins with the essential preliminary matter, the *Definitions* (ὅροι), *Postulates* (αἰτήματα), and *Common Notions* (κοιναὶ ἔννοιαι). The *Common Notions* are what we know as *Axioms*, for which Aristotle has the alternative names 'common (things)', 'common opinions'.

Many of the *Definitions* are open to criticism. Two at least seem to be original, namely those of (1) a straight line and (2) a plane. Unsatisfactory as these are, they seem to be capable of a simple explanation. Plato had defined a straight line as 'that of which the middle covers the ends' (i.e. to an eye placed at one end and looking along the line), and Euclid's 'line which lies evenly with the points on itself' may well be an attempt to express Plato's idea in terms excluding any appeal to sight; so also with the definition of a plane. But most of the definitions were probably taken from earlier text-books; this is no doubt the reason why some were included which are never used in the *Elements*, e.g. the definitions of *oblong*,

rhombus, and *rhomboid*. A square and different kinds of triangles are defined, but there is no definition of a parallelogram. After the existence of a parallelogram is proved (in I. 33), it is first called a *parallelogrammic area* (i.e. an area contained by parallel straight lines), and then (I. 35) the name is shortened to *parallelogram*. To the definition of the diameter of a circle is added the statement that 'such a straight line also bisects the circle' (a discovery attributed to Thales); no doubt this is really a theorem, but the addition was necessary in order to justify the definition of a *semi*-circle immediately following, namely 'the figure contained by the diameter and the circumference cut off by it.'

The five *Postulates* are more important, for they embody the distinctive principles of Euclidean geometry. The first three are commonly regarded as the postulates of *construction*, since they assert the possibility (1) of drawing a straight line joining two given points, (2) of producing a straight line continuously in either direction, (3) of describing a circle with a given centre and 'distance'. Euclid here postulates the existence of real (mathematical) straight lines and circles, of which the straight lines and circles that can in practice be drawn are only imperfect illustrations. Postulates 1 and 2 also imply that the straight line in the first case and the produced portion in the second case are *unique*; in other words, they imply that 'two straight lines cannot enclose a space' and that 'two straight lines cannot have a common segment', and they obviate the necessity for a separate statement of these facts (the reference to the former 'axiom' in I. 4 is in fact interpolated).

Postulates 4 (that all right angles are equal) and 5 (the Parallel-Postulate) might seem to be of an entirely different character. Euclid, however, having to lay down *some* postulate as a basis for a theory of parallels, actually formulated a postulate which also supplies a criterion indispensable in constructions, namely a means of knowing whether two straight lines drawn in a figure will or will not meet if produced. This is one of the advantages of Euclid's Postulate as compared with other equivalents such as Playfair's; and Euclid actually employs it for this purpose as early as I. 44. No doubt Postulate 4 about right angles is often classed as a theorem. But, if we are to prove it, we can only do so by applying one pair of adjacent right angles to another pair; and this method would not be valid except on the assumption of the *invariability of figures*, which would, therefore, strictly speaking, have to be asserted as an antecedent postulate. Euclid had, in any case, to place Postulate 4 before Postulate 5, because Postulate 5 would be no criterion at all unless right angles were determinate magnitudes.

Of the *Common Notions* it is probable that only five (at most) are genuine, the first three and two others, namely 'Things which coincide with [lit. 'fit on'] one another are equal to one another' (4), and 'The whole is greater than the part' (5). The objection to (4) is that its subject-matter belongs to a special science, namely geometry, whereas, according to Aristotle, 'axioms' or 'common notions' are general truths common to all sciences (e.g., if equals be subtracted from equals, the remainders are equal). Neither of the two supposed Axioms 4 and 5 seems to be quoted in terms by Euclid himself; thus in I. 4, where he might have quoted the former, he says simply, 'The base *BC* will coincide with the base *EF* and will be equal to it', without referring to any axiom. It seems probable, therefore, that these two Common Notions, though recognized by Proclus, were generalizations from particular inferences found in Euclid, and were inserted after his time.

The propositions of Book I fall into three distinct groups. The first, consisting of Propositions 1–26, deals mainly with triangles, their construction, and their properties in the sense of the relation of their parts, the sides and angles, to one another, including the three congruence-theorems; it also treats of two intersecting straight lines making 'vertically opposite' angles, and one straight line standing on another and making 'adjacent' angles; and it contains a few simple problems of construction, the drawing of perpendiculars, and the bisection of a given angle and of a given straight line. The second group, beginning with I. 27, establishes the theory of parallels, and leads up to the proposition that the sum of the three angles of any triangle is equal to two right angles (32). The third group, beginning with I. 33, 34, which introduce the parallelogram for the first time, deals generally with parallelograms, triangles, and squares with reference to their areas. Propositions 44, 45 are of the greatest importance, being the first cases of the Pythagorean method of 'application of areas': to apply to a given straight line, in a given rectilineal angle, a parallelogram equal to a given triangle (or rectilineal figure); the solution depends on I. 43 proving that the 'complements' of the parallelograms about the diameter of a parallelogram are equal in area, and is equivalent to the algebraical operation of dividing the product of two quantities by a third. I. 46 shows how to construct a square on any given straight line as side, and I. 47 is the great Pythagorean theorem of the square on the hypotenuse of a right-angled triangle. With a converse to the latter theorem the Book ends (I. 48).

Book II is a continuation of the third section of Book I relating to the transformation of areas. It deals mainly with *rectangles* (which appear for the first time) and squares, and not with parallelograms in general, and it shows the equality of sums of rectangles and squares to other such sums. Much use is made of the *gnomon*; this is defined (Def. 2) with reference to any parallelogram, but the gnomons actually used are those belonging to squares. The whole Book is part of the *geometrical algebra* which, with the Greeks, had to take the place of our algebra. The first ten propositions of Book II give the equivalent of the following algebraical identities:

1. $a(b+c+d+\ldots) = ab+ac+ad+\ldots$
2. $(a+b)a+(a+b)b = (a+b)^2$,
3. $(a+b)a = ab+a^2$,
4. $(a+b)^2 = a^2+b^2+2ab$,
5. $ab+\{\frac{1}{2}(a+b)-b\}^2 = \{\frac{1}{2}(a+b)\}^2$,
   or $(\alpha+\beta)(\alpha-\beta)+\beta^2 = \alpha^2$,
6. $(2a+b)b+a^2 = (a+b)^2$,
   or $(\alpha+\beta)(\beta-\alpha)+\alpha^2 = \beta^2$,
7. $(a+b)^2+a^2 = 2(a+b)a+b^2$,
   or $\alpha^2+\beta^2 = 2\alpha\beta+(\alpha-\beta)^2$,
8. $4(a+b)a+b^2 = \{(a+b)+a\}^2$,
   or $4\alpha\beta+(\alpha-\beta)^2 = (\alpha+\beta)^2$,
9. $a^2+b^2 = 2[\{\frac{1}{2}(a+b)\}^2+\{\frac{1}{2}(a+b)-b\}^2]$,
   or $(\alpha+\beta)^2+(\alpha-\beta)^2 = 2(\alpha^2+\beta^2)$,
10. $(2a+b)^2+b^2 = 2\{a^2+(a+b)^2\}$,
   or $(\alpha+\beta)^2+(\beta-\alpha)^2 = 2(\alpha^2+\beta^2)$.

As we have seen (pp. 103–4), Propositions 5 and 6 enable us to solve geometrically the equivalent of the quadratic equations

$$(1)\quad ax-x^2 = b^2,\ \text{or}\begin{cases} x+y = a, \\ xy = b^2, \end{cases}$$

and

$$(2)\quad ax+x^2 = b^2,\ \text{or}\begin{cases} y-x = a, \\ xy = b^2. \end{cases}$$

The procedure is geometrical throughout, and the various areas in all the Propositions 1–8 are actually shown in the figures.

Propositions 9 and 10 were (as we have seen, p. 57 above) used to solve a problem in numbers, namely that of finding any number of successive pairs of integral numbers ('side-' and 'diameter-' numbers) satisfying the equations

$$2x^2-y^2 = \pm 1.$$

Of the remaining propositions, II. 11 and II. 14 give the geometrical equivalent of the solution of the quadratics

$$x^2+ax = a^2$$

and

$$x^2 = ab,$$

while Propositions 12 and 13 prove for any triangle with sides a, b, c, the equivalent of the formula

$$a^2 = b^2+c^2-2bc\cos A.$$

It is worth noting that, just as Book I seems designed to lead up to the Pythagorean proposition I. 47 and its converse, so Book II gives, in its last two propositions but one, a generalization of that theorem with any triangle taking the place of the right-angled triangle.

Book III is on the geometry of the circle, including the relations between circles cutting or touching one another. It begins with definitions. 'Equal circles' are defined as circles with equal diameters or radii (the Greeks had no single word for 'radius'; they called it 'the (straight line) from the centre', ἡ ἐκ τοῦ κέντρου); if this 'definition' were proved, it could only be by superposition. A 'tangent' is defined (Def. 2) as 'a straight line which meets a circle but, if produced, does not cut it'. A chord is simply 'a straight line in a circle', and chords are equally distant, or more or less distant, from the centre, according as the perpendiculars on them from the centre are equal, greater, or less (Defs. 4, 5). Euclid defines not only a segment of a circle and the 'angle in a segment' (Defs. 6, 8), but also the 'angle *of* a segment' (Def. 7). The last-named definition, as well as the part of Proposition 16 about the 'angle *of* a semicircle', are the last survivals in Greek geometry of the 'angle *of* a segment' (the 'mixed' angle made by the curve with the base of the segment at either end); these survivals show Euclid's almost excessive respect for tradition, the 'angle' in question being of no practical use in demonstrations. The last definitions are those of a *sector* of a circle and of 'similar segments'; the word for *sector*, τομεύς, is said to have been suggested by the shape of the 'shoemaker's knife' (σκυτοτομικὸς τομεύς). The definition of 'similar segments' assumes provisionally (pending the proof in III. 21) that the angle in a segment is one and the same at whatever point on the circumference it is formed.

The propositions of Book III may be roughly classified thus. Central and chord properties account for six propositions (1, 3, 4, 9, 14, 15). Three propositions throw light on the form of the circle (2, showing that it is everywhere concave towards the centre, and 7, 8, comparing the respective lengths of all straight lines drawn to the circumference from a single point (other than the centre), internal or external. Propositions 5, 6, 10, 11, 13 and the interpolated Proposition 12 deal with two circles cutting or touching one another. Tangent properties, including the drawing of a tangent, occupy Propositions 16–19; it is in 16 that we have the survival of the 'angle *of* a semicircle' and of its complement, the 'angle' between the

curve and the tangent at the extremity of the diameter, the latter angle (afterwards called the κερατοειδής or 'hornlike' angle) being proved to be less than any rectilineal angle. These 'mixed' angles, occurring here and in Proposition 31 only, appear no more in serious Greek geometry, though controversy about the nature of the 'hornlike angle' went on in the works of commentators down to Clavius, Peletarius, Vieta, Galilei, and Wallis. Propositions 20–34 are concerned with segments, angles in segments and at the centre, &c. The Book ends with three important propositions (35–7) to the effect that, 'given a circle and any point O internal or external to it, if any straight line through O meet the circle in P, Q, the rectangle $PO\,.\,OQ$ is constant and, in the case where O is external to the circle, is equal to the square on the tangent to the circle from O.

Book IV continues the geometry of the circle, with special reference to the problems of inscribing and circumscribing to the circle certain rectilineal figures which can be so inscribed and circumscribed by means of the geometry of the straight line and circle only, namely, a triangle equiangular with a given triangle (2, 3), a square (6, 7), a regular pentagon (11, 12), a regular hexagon (15), and a regular polygon with fifteen sides (16), and the corresponding problems of inscribing or circumscribing a circle to a triangle (4, 5), a square (8, 9), and the other figures mentioned (13, 14, 15 Por.). IV. 10 is the important Pythagorean proposition, used in the construction of a regular pentagon, 'To construct an isosceles triangle having each of the angles at the base double of the remaining angle', which again depends on the Pythagorean proposition (II. 11) showing how to divide a given straight line in extreme and mean ratio. The regular fifteen-sided figure (Prop. 16) was found useful in astronomy, the obliquity of the ecliptic being taken to be about 24° or one-fifteenth of 360°. The whole of the Book seems to be unquestionably Pythagorean.

Book V expounds the new theory of proportion applicable to incommensurable as well as commensurable magnitudes, and to magnitudes of every kind (straight lines, angles, areas, volumes, numbers, times, &c.). Greek mathematics can boast of no finer discovery than this theory, due to Eudoxus, which first put on a sound footing so much of geometry as depended on the use of proportions. The scholiast who attributes the discovery of the theory to Eudoxus is equally clear that the actual arrangement and sequence of Book V is due to Euclid himself. The ordering of the propositions and the development of the proofs are indeed masterly and worthy of Euclid; as Barrow said, 'there is nothing in the whole body of the Elements of a more subtile invention, nothing more solidly established and more accurately handled, than the doctrine of proportionals'.

The Definitions of Book V are naturally of supreme importance. The definition (3) of ratio as 'a sort of relation (ποιὰ σχέσις) in respect of size (πηλικότης) between two magnitudes of the same kind' tells us little, certainly; but Definition 4 ('Magnitudes are said to have a ratio to one another which are capable when multiplied of exceeding one another') makes amends, for not only does it show that the magnitudes must be of the same kind, but, while including incommensurable as well as commensurable magnitudes, it *excludes* the relation between a finite magnitude and a magnitude of the same kind which is infinitely great or infinitely small; it is also practically equivalent to the 'Axiom of Archimedes' (so-called), which lies at the

root of the method of exhaustion. Most important of all is the fundamental definition (5) of magnitudes which are in the same ratio: 'Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or are alike less than, the latter equimultiples taken in corresponding order.' Perhaps the greatest tribute to this wonderful definition is its adoption by Weierstrass as a definition of equal numbers. For a most attractive explanation showing its exact significance and its absolute sufficiency the reader should refer to De Morgan's articles on Ratio and Proportion in the *Penny Cyclopaedia* (vol. xix, 1841), largely reproduced in *The Thirteen Books of Euclid's Elements* (vol. ii, pp. 116–24). Euclid adds (7) a definition of 'greater ratio': 'When of the equimultiples the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a *greater ratio* to the second than the third has to the fourth'; here Euclid takes (possibly for brevity) only one criterion for greater ratio, the other possible criterion being that, while the multiple of the first is *equal* to that of the second, the multiple of the third is *less* than that of the fourth. A proportion may be in three or four terms (Defs. 8, 9, 10); 'corresponding' or 'homologous' terms mean antecedents in relation to antecedents and consequents in relation to consequents (11). Definitions 12–16 explain the terms used for the transformation of ratios: (α) ἐναλλάξ, *alternando*, transforms the proportion $a:b = c:d$ into $a:c = b:d$; (β) *inversion* (ἀνάπαλιν, *inversely*) turns the ratio $a:b$ into $b:a$; (γ) *composition*, σύνθεσις (συνθέντι, lit. 'to one who has compounded', = *componendo*) turns $a:b$ into $(a+b):b$; (δ) *separation*, διαίρεσις (διελόντι, lit. 'to one who has separated', = *separando*) turns $a:b$ into $(a-b):b$; (ε) *conversion*, ἀναστροφή (ἀναστρέψαντι = *convertendo*) turns $a:b$ into $a:(a-b)$. Lastly, we have definitions (17, 18) of *ex aequali* (sc. *distantia*), δι' ἴσου, and *ex aequali* 'in disturbed proportion' (δι' ἴσου ἐν τεταραγμένῃ ἀναλογίᾳ); the first infers from $a:b=A:B$ and $b:c=B:C$ that $a:c=A:C$, and the second infers from $a:b=B:C$ and $b:c=A:B$ that $a:c=A:C$.

As the content of the wonderful Book V is too little known, it is worth while to summarize it with the aid of modern notation. In the summary the letters $a, b, c\ldots$ will mean *magnitudes* in general and the letters $m, n, p\ldots$ integral *numbers*; thus ma, mb are equimultiples of a, b.

The first six propositions are arithmetical theorems about multiples and equimultiples.

$$\begin{cases} 1. & ma+mb+mc+\ldots = m(a+b+c+\ldots). \\ 5. & ma-mb = m(a-b). \end{cases}$$

$$\begin{cases} 2. & ma+na+pa+\ldots = (m+n+p+\ldots)a. \\ 6. & ma-na = (m-n)a. \end{cases}$$

3. Equimultiples of equimultiples are themselves equimultiples.

4. If $a:b=c:d$, then $ma:nb=mc:nd$; or the equimultiples in Def. 5 are themselves proportionals.

All these propositions except (4) are proved by separating the multiples used into their units. (4) is proved by taking equimultiples of the equimultiples, namely pma and pmc of ma, mc, and qnb, qnd of nb, nd. Then, by 3, the new equimultiples are equimultiples of a, c and b, d respectively. Since $a:b=c:d$, the new equimultiples satisfy the criterion of Def. 5, whence conversely

$$ma:nb=mc:nd.$$

7, 9. If $a=b$, then $\left.\begin{array}{l} a:c=b:c \\ \text{and } c:a=c:b \end{array}\right\}$; and conversely.

8, 10. If $a>b$, then $\left.\begin{array}{l} a:c>b:c \\ \text{and } c:b>c:a \end{array}\right\}$; and conversely.

7, 8 are proved by using Defs. 5 and 7, and the converses are proved by *reductio ad absurdum*.

11. If $a:b=c:d$,
and $c:d=e:f$,
then $a:b=e:f$.

12. If $a:b=c:d=e:f\ldots$
then $a:b=(a+c+e+\ldots):(b+d+f+\ldots)$.

13. If $a:b=c:d$,
but $c:d>e:f$,
then $a:b>e:f$.

14. If $a:b=c:d$,
then, according as $a \gtreqless c$, $b \gtreqless d$;
that is, the criterion of Def. 5 is true if, instead of equimultiples, we take *once* the magnitudes respectively.

15. $a:b=ma:mb$. This follows from 12.

16–18 prove the legitimacy of transforming a proportion *alternando*, *separando*, *componendo* respectively; that is, they prove that, if the original proportion is true, the transformed proportion is also true.

19. If $a:b=c:d$,
then $(a-c):(b-d)=a:b$.

The transformation of a proportion by *inversion* is not given, probably because it is obvious from Def. 5; transformation by *conversion* is not given either, but it follows, as 19 does, by using 17 combined with 16.

20–3 establish the truth of inferences from two proportions *ex aequali* and *ex aequali* 'in disturbed proportion' respectively, 20 being preliminary to 22 and 21 to 23; i.e. it is proved

(22) that, if $a : b = d : e$,
and $b : c = e : f$,
then, *ex aequali*, $a : c = d : f$,
and (23) that, if $a : b = e : f$,
and $b : c = d : e$,
then, *ex aequali* 'in disturbed proportion', $a : c = d : f$.

The Book concludes with

24. If $a : c = d : f$,
and $b : c = e : f$,
then $(a+b) : c = (d+e) : f$.

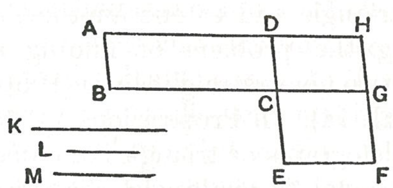
25. If $a : b = c : d$, and of the four terms a is the greatest (so that d is the least), then

$$a+d > b+c.$$

Some slight defects are found in the Book as it has reached us, and perhaps, therefore, it never received the final touches from Euclid's hand; but these defects can all be corrected without much difficulty, as Simson showed in his admirable edition. M. J. M. Hill has gone further and, after long and unremitting labour, has recently contributed valuable papers to the *Mathematical Gazette* supplementing Euclid and making of his system a consistent and well-rounded whole.

Book VI applies the general theory of proportion set out in Book V to plane geometry. The first proposition, proving that triangles and parallelograms of the same height are respectively as their bases, and the last (33), to the effect that in equal circles angles at the centre or at the circumference respectively are as the arcs on which they stand, both use the method of equimultiples and apply the test of proportion laid down in V. Def. 5. The fundamental proposition (2) that two sides of a triangle cut by any parallel to the third side are divided proportionally, and the converse, gives the means of solving the problems of cutting off from a straight line a prescribed part (9), of cutting a given straight line proportionally to a given divided straight line (10), of finding a third proportional to two straight lines (11), and a fourth proportional to three (12). Proposition 3 proves that the internal bisector of an angle of a triangle cuts the opposite side into parts which have the same ratio as the sides containing the angle, and the converse. Next come propositions showing the alternative conditions for the similarity of two triangles, namely equality of all the angles respectively (4), proportionality of pairs of sides in order (5), equality of one angle in each with proportionality of the sides containing the equal angles (6), and (the 'ambiguous case') equality of one angle in each and proportionality of the sides containing other angles (7). Proposition 8 proves that the perpendicular from the right angle in a right-angled triangle to the opposite side divides the triangle into two triangles which are similar to the original triangle and to one another, a proposition used in solving the problem of finding a mean proportional between two given straight lines (Prop. 13, the Book VI version of II. 14). In Propositions 14, 15 Euclid proves that, in parallelograms or triangles of equal area which have one angle equal to one angle, the sides about the equal angles are reciprocally proportional, and the converse. It is then proved (16), by means of 14, that, if four straight lines are proportional, the rectangle contained by the extremes is equal to that contained by the means, and conversely; Proposition 17 contains the particular case of three proportional straight lines, where the rectangle contained by the extremes is equal to the square on the mean. Propositions 18–22 deal with similar rectilineal figures; 19 (with Porism) and 20 are specially important, proving that similar triangles, and similar polygons generally, are to one another in the duplicate ratio of corresponding sides, and that, if three straight lines are proportional, then, as the first is to the third, so is the figure described on the first to the similar figure similarly described on the second.

Proposition 23 (equiangular parallelograms have to one another the ratio compounded of the ratios of their sides) is highly important in itself, and also because it introduces us to the method of compounding, i.e. multiplying, ratios, a practical method of very wide application in Greek geometry. Euclid has never defined 'compound ratio' or the 'compounding' of ratios; the meaning of 'compound ratio' and the method of compounding are made clear by this proposition. The equiangular parallelograms are placed so that two equal angles as BCD, GCE are vertically opposite at C, or BCG, ECD are straight lines. Complete the parallelogram $DCGH$. The ratio 'compounded of the ratios of the sides' of the parallelograms AC, CF is the ratio compounded of the ratios $BC : CG$ and $DC : CE$, and is obtained thus. Take any straight line K and find another, L, such that



$$BC : CG = K : L.$$

Again, find a straight line, M, such that

$$DC : CE = L : M.$$

Then the ratio 'compounded of the ratios of the sides' is equal to the ratio compounded of the ratios $K : L$ and $L : M$, that is, the ratio $K : M$.

But (VI. 1) $\quad (ABCD) : (DCGH) = BC : CG,$

$$= K : L;$$

and $\quad (DCGH) : (CEFG) = DC : CE,$

$$= L : M.$$

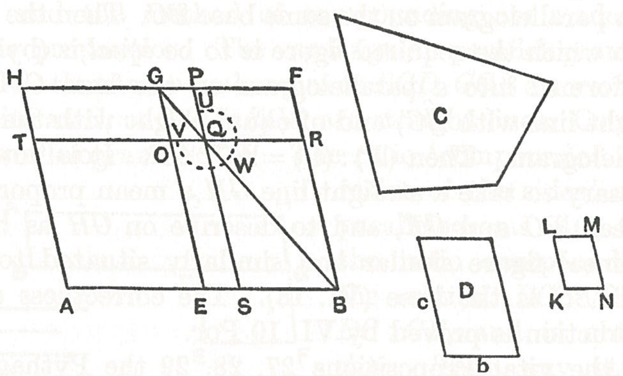
Therefore, *ex aequali* (V. 22)

$$(ABCD) : (CEFG) = K : M.$$

The important Proposition 25 (to construct a rectilineal figure similar to one, and equal to another, rectilineal figure) is one of the famous propositions attributed to Pythagoras; it is doubtless Pythagorean, since it employs precisely the problems in 'application of areas' contained in Eucl. I. 44, 45. The given figure (P, say) to which the required figure is to be similar is transformed (by I. 44) into a parallelogram on the same base BC. Then the figure (Q) to which the required figure is to be *equal* is (by I. 45) transformed into a parallelogram on the base CF (in a straight line with BC) and of equal height with the other parallelogram. Then $(P) : (Q) = BC : CF$. It is now only necessary to take a straight line GH a mean proportional between BC and CF, and to describe on GH as base a rectilineal figure similar and similarly situated to P in which BC is the base (VI. 18). The correctness of the construction is proved by VI. 19 Por.

In the vital Propositions 27, 28, 29 the Pythagorean *application of areas* appears in its most general form, equivalent to the geometrical solution of the most general form of quadratic equation where that equation has a real and positive root. The method is fundamental in Greek geometry; it is, for instance, the foundation of Euclid's Book X (on irrationals) and of the whole treatment of conic sections by Apollonius of Perga. The problems of Propositions 28, 29 are thus enunciated: 'To a given straight line to apply a parallelogram equal to a given rectilineal figure and *deficient* (or *exceeding*) by a parallelogrammic figure similar to a given parallelogram'; and Proposition 27 proves the διορισμός, or determination of the condition of possibility of solution, in the case of *deficiency* (28): 'The given rectilineal figure must not be greater than the parallelogram described on the half of the straight line and similar to the defect.'

We will first examine the problem of Proposition 28. We are already familiar with the notion of applying a parallelogram to a straight line AB so that it *falls short* by a certain other parallelogram. Suppose that D is the given parallelogram to which the *defect* has to be similar. Bisect AB at E, and on the half EB describe the parallelo-



gram $GEBF$ similar and similarly situated to D. Draw the diagonal GB and complete the parallelogram $HABF$.

Through any point T on HA draw $TOQR$ parallel to AB meeting GE, GB and FB in O, Q, R respectively, and through Q draw PQS parallel to HA or GE. Then $TASQ$ is a parallelogram applied to AB but *falling short* by a parallelogram ($QSBR$) which is similar and similarly situated to EF (24) and therefore to D (21). (In the same way, if T had been on HA *produced* and TR had met GB *produced*, we should have had a parallelogram applied to AB but *exceeding* by a parallelogram similar and similarly situated to D.)

Now consider the parallelogram AQ applied to AB but falling short by SR which is similar and similarly situated to D. Since $(AO) = (ER)$ and $(OS) = (QF)$, the parallelogram AQ is equal to the gnomon UWV, and the problem is therefore to find the gnomon UWV such that its area is equal to that of the given rectilineal figure C. And clearly the gnomon cannot be greater than the parallelogram EF. Therefore C must not be greater than EF; and this is the διορισμός proved in 27.

Since now the gnomon UWV has to be equal to C, the parallelogram $GOPQ$ must be equal to the difference between (EF) and C. Hence we have merely to draw in the angle FGE the parallelogram $GOQP$ equal to $(EF)-C$ and similar and similarly situated to D. Euclid does in fact describe the parallelogram $LKNM$ equal to $(EF)-C$ and similar and similarly situated to D, and then draws $GOQP$ equal to it. The problem is then solved, $TASQ$ being the required parallelogram applied to AB and falling short in the manner required.

We will now show that Euclid's geometrical solution corresponds exactly to our algebraical solution of a quadratic equation. Let $AB = a$, and let $b : c$ be the ratios of the sides of D as shown in the figure. We may take for x, the unknown quantity, either of the sides of the *defect SR*. If $SB = x$, $QS = \frac{c}{b}x$. Then, if m is a certain constant (actually the sine of an angle of one of the parallelograms), the area (AQ) is $m.AS.SQ$ or $m(a-x)\frac{c}{b}x$. This is equal to C, and the equation to be solved is

$$ax - x^2 = \frac{b}{c}\cdot\frac{C}{m}.$$

To solve this, we change the sign throughout and complete the square on the left-hand side; thus

$$(\tfrac{1}{2}a - x)^2 = \tfrac{1}{4}a^2 - \frac{b}{c}\cdot\frac{C}{m},$$

and $$x = \tfrac{1}{2}a \mp \sqrt{\left(\tfrac{1}{4}a^2 - \frac{b}{c}\cdot\frac{C}{m}\right)}.$$

Now Euclid actually constructs the parallelogram $GOPQ$. The area of this is

$$m.GO.OQ = m.\frac{c}{b}.OQ^2 = m.\frac{c}{b}(\tfrac{1}{2}a - x)^2;$$

that is, Euclid in effect finds $(\frac{1}{2}a - x)^2$ just as we do. The solution in the figure corresponds to the *negative* sign before the radical

$$x = \tfrac{1}{2}a - \sqrt{\left(\tfrac{1}{4}a^2 - \frac{b}{c}\cdot\frac{C}{m}\right)};$$

but Euclid was, of course, aware that there are two solutions, and how he could exhibit the second in the figure.

For a real solution C must not be greater than $m(\frac{1}{2}a)^2 . \frac{c}{b}$, which is the area of (EF). This is what is proved in Euclid's Proposition 27.

The solution of Proposition 29 is similar *mutatis mutandis*,

but no διορισμός is necessary, a solution being always possible.

Proposition 30 uses 29 for the purpose of dividing a straight line 'in extreme and mean ratio'. Proposition 31 extends the theorem of I. 47, showing that that theorem is true not only of squares but of three similar plane figures (of whatever shape) described upon the three sides of the right-angled triangle and similarly situated with reference to the sides.

Except in the respect that it is based on the new theory of proportion, Book VI does not appear to contain any matter that was not known before Euclid's time. The extension of I. 47 is assumed by Hippocrates of Chios for semicircles described on the sides of a right-angled triangle as diameters.

Books VII, VIII, IX are arithmetical in the Greek sense, that is to say, they deal mainly with the nature and properties of (integral) numbers. Book VII begins with definitions, including those of a *unit*, a *number*, and the varieties of numbers, *even*, *odd*, *even-times even*, *even-times odd*, *odd-times odd*, *prime*, *prime to one another*, *composite*, *composite to one another*, *plane*, *solid*, *square*, *cube*, *similar plane* and *similar solid* numbers, and a *perfect* number. There are also definitions of terms employed in the numerical theory of proportion, namely *a part* (= an aliquot part or submultiple), *parts* (= a certain number of such parts, equivalent to a proper fraction), *multiply*; and we have finally the definition of four proportional numbers, stating that 'numbers are proportional when the first is the same multiple, the same part, or the same parts, of the second that the third is of the fourth', i.e. numbers a, b, c, d are proportional if, when $a = \frac{m}{n}b$, $c = \frac{m}{n}d$, where m, n are any integers (though the definition does not in terms cover the case where $m > n$). The mode of presentation is geometrical in the sense that numbers are throughout represented by straight lines and not by numerical signs.

The propositions of Book VII fall into four main groups. Propositions 1–3 give the method of finding the greatest common measure of two or three numbers in essentially the same form as that in which our text-books have it; the test for two numbers being prime to one another (namely, that no remainder measures the preceding divisor till 1 is reached) comes in the first proposition. Propositions 4–19 set out the numerical theory of proportion; 4–10 are preliminary, dealing with numbers which are 'a part' or 'parts' of other numbers, or 'the same part' or 'parts' of other numbers respectively, and so connecting the theory with the definition of proportionals. 11 and 13 are transformations of proportions corresponding to V. 19 and V. 16, while 12 corresponds to V. 12 and 14 to V. 22 (the *ex aequali* proposition).

Proposition 15 proves that, if $1 : m = a : ma$ ('if the third number measures the fourth the same number of times that the unit measures the second'), then alternately

$$1 : a = m : ma.$$

This result is used (16) to prove that $ab = ba$, or, in other words, that the order of multiplication is indifferent. Two simple propositions (17, 18), based on 16, namely that $b : c = ab : ac$ and $a : b = ac : bc$, lead to the important Proposition 19 that, if $a : b = c : d$, then $ad = bc$, and conversely, which corresponds to VI. 16 for straight lines.

Propositions 20, 21 about 'the least numbers of those which have the same ratio with them' prove that, if m, n are such numbers, and a, b any others in the same ratio, m measures a the same number of times that n measures b, and that numbers prime to one another are the least of those which have the same ratio with them. These propositions lead up to propositions (22–32) about numbers prime to one another, prime numbers, and composite numbers; of which we may mention: (24) if two numbers be prime to any number, their product will also be prime to the same number; (28) if two numbers be prime to one another, their sum will be prime to each of them, and, if the sum be prime to either, the original numbers will be prime to one another; (30) if any prime number measures the product of two numbers, it will measure one of the two; (32) any number either is prime or is measured by some prime number.

Propositions 33–9 are directed to finding the least common multiple of two or three numbers; 33 is preliminary, using the G. C. M. for the purpose of solving the problem, 'Given as many numbers as we please, to find the least of those which have the same ratio with them'.

In Book VII Euclid was probably following earlier models, while making improvements of his own. Propositions corresponding to VII. 20, 22, 33 are presupposed in the fragment of Archytas already referred to (pp. 136–7).

Book VIII deals largely with numbers in continued 'proportion', i.e. in geometrical progression. If we denote the terms by $a^n, a^{n-1}b, a^{n-2}b^2 \ldots ab^{n-1}, b^n$, we learn that, if a^n, b^n are prime to one another, the terms of the series are the least of those which have the same ratio with them, and vice versa (1, 3); 2 shows how to find the series when the ratio $a : b$ is given in its lowest terms. If a^n does not measure $a^{n-1}b$, no term measures any other, but if a^n measures b^n, it measures $a^{n-1}b$ (6, 7). According as a^2 does or does not measure b^2, and according as a^3 does or does not measure b^3, a does or does not measure b, and vice versa (14–17). If a, b, c are in geometrical progression, so are a^2, b^2, c^2 ... and a^3, b^3, c^3 ... respectively (13).

Given any number of ratios between numbers as $a:b$, $c:d$ ..., Proposition 4 shows how to find a series p, q, r ... in the least possible terms such that

$$p:q=a:b,\ q:r=c:d, \dots .$$

This is done by finding the least common measure first of b, c and then of other pairs of numbers as required. This proposition enables us to compound any number of ratios between numbers in the same way as ratios between straight lines are compounded in VI. 23; the proposition (5) corresponding to VI. 23 then follows to the effect that plane numbers have to one another the ratio compounded of the ratios of their sides.

Propositions 8–12 and 18–21 deal with the interpolation of geometric means between numbers. If $a:b=e:f$, and there are n geometric means between a and b, there are n geometric means between e and f (8). If a^n, $a^{n-1}b$, $a^{n-2}b^2$ ... ab^{n-1}, b^n is a G. P. of $n+1$ terms, so that there are $n-1$ geometric means between a^n and b^n, there are the same number of geometric means between 1, a^n and between 1, b^n respectively (9); and conversely, if 1, a, a^2 ... a^n, and 1, b, b^2 ... b^n are series in geometrical progression, there are the same number $(n-1)$ of geometric means between a^n, b^n as there are between 1, a^n and between 1, b^n respectively (10). There is one mean proportional number between two square numbers (11) and between two similar plane numbers (18), and conversely, if there is one mean proportional number between two numbers, the numbers are similar plane numbers (20); there are two geometric means between two cube numbers (12) and between two similar solid numbers (19), and conversely, if there are two geometric means between two numbers, the numbers are similar solid numbers (21). These propositions are stated for square and cube numbers by Plato in the *Timaeus*, and Nicomachus calls them 'Platonic' accordingly. Lastly, similar plane numbers have the same ratio as a square has to a square (26), and similar solid numbers have the same ratio as a cube has to a cube (27).

Book IX begins with some simple propositions such as the following: the product of two similar plane numbers is a square (1), and, if the product of two numbers is a square number, the numbers are similar plane numbers (2); the product of two equal or unequal cubes is a cube (3, 4); if a^3B is a cube, B is a cube (5), and if A^2 is a cube, A is a cube (6). Propositions 8–13 prove certain relations between the terms of a series in geometrical progression in which 1 is the first term. If the series is 1, a, b, c, ... k, then (9), if a is a square (or a cube), all the succeeding terms are squares (or cubes); if a is not a square, the only squares in the series are the term following a, namely b, and all alternate terms after b; if a is not a cube, the only cubes in the series are the fourth term (c), the seventh, tenth, &c., terms (leaving out two terms throughout); the seventh, thirteenth, &c., terms (leaving out five terms each time) are both square and cube (8, 10). The interesting theorem follows (11 and Porism) that, if 1, a_1, a_2 ... a_n are terms in geometrical progression, and a_r, a_n are any two terms, a_r being less than a_n, then a_r will measure a_n, and $a_n = a_r \cdot a_{n-r}$; this is of course equivalent to the formula $a^{m+n} = a^m \cdot a^n$. It is next proved (12, 13) that, if 1, a, b, c ... k are numbers in geometrical progression, and k is measured by any primes, a is measured by the same; and if a is prime, k will not be measured by any numbers except those which occur in the series. Proposition 14 is the equivalent of the important theorem that *a number can only be resolved into prime factors in one way*. Propositions 16–19 deal with the conditions under which it is possible, or impossible, that there should be an integral third proportional to two, or an integral fourth proportional to three, given numbers. Next, by a proof which is the same as that usually given in our algebras, Euclid proves (20) that *the number of prime numbers is infinite.* After some easy propositions (21–34) about odd, even, even-times odd, and even-times even numbers respectively, Euclid gives in the last two propositions of the Book an elegant summation of a G. P. of n terms (35), and a proof of the criterion for the formation of 'perfect' numbers (36).

The summation of the G. P. amounts to the following. Suppose $a_1, a_2, a_3 \ldots a_{n+1}$ to be $n+1$ terms in G. P.

Then $$\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}} = \ldots = \frac{a_2}{a_1},$$

and, *separando*, $$\frac{a_{n+1}-a_n}{a_n} = \frac{a_n-a_{n-1}}{a_{n-1}} = \ldots = \frac{a_2-a_1}{a_1}.$$

Adding the antecedents and the consequents, we have (VII. 12)

$$\frac{a_{n+1}-a_1}{a_n+a_{n-1}+\ldots+a_2+a_1} = \frac{a_2-a_1}{a_1}$$

which gives $a_n+a_{n-1}+\ldots+a_2+a_1$ or $\Sigma_1^n a$.

In Proposition 36 Euclid proves that, if the sum of any number of terms of the series $1, 2, 2^2 \ldots 2^n$ is prime, the product of the said sum and of the last term, or

$$(1+2+2^2+\ldots+2^n)\,2^n$$

is a *perfect* number, i.e. is equal to the sum of all its factors.

In the arithmetical Books all numbers are, as we said,

represented by straight lines. This applies throughout, whether the numbers are linear, plane, or solid, or any other kinds of numbers: thus a product of two or three numbers is represented, not by a rectangle or a solid, but by a straight line.

Book X is perhaps the most remarkable, as it is certainly the most finished, of all the Books in the *Elements*. It deals with irrationals, by which must be understood, in general, *straight lines* which are irrational in relation to any particular straight line assumed as rational; and it investigates every possible variety of straight line corresponding to what we should express in algebra by $\sqrt{(\sqrt{a}\pm\sqrt{b})}$, where $\sqrt{a}$ and $\sqrt{b}$ are surds and incommensurable with one another. The subject did not originate with Euclid. We know that not only the fundamental proposition X. 9 (proving that squares which have not to one another the ratio of a square number to a square number have their sides incommensurable in length, and vice versa), but also a large part of the further development of the subject, was due to Theaetetus. But, as Pappus says, in a commentary partly extant in Arabic, Euclid systematized the theory, making precise the definitions of rational and irrational magnitudes, setting out a number of orders of irrational magnitudes and exhibiting their whole extent.

To begin with the definitions. 'Commensurable' magnitudes can be measured by one and the same measure; 'incommensurable' magnitudes have no common measure (1). Straight lines incommensurable in length may be 'commensurable in square' or 'incommensurable in square' according as the squares on them can or cannot be measured by one and the same area (2). Given a straight line which we agree to call rational, Euclid regards as rational not only any straight line commensurable in length with the given straight line, but also any straight line commensurable with it in square though not in length; if, however, a straight line is commensurable neither in length nor in square with the given rational straight line, it is irrational (3). On the other hand, while the square on the straight line assumed as rational is rational, any area incommensurable with it is irrational (4). Thus, if ρ is a straight line assumed as rational, not only is $k\rho$ rational, but also $\sqrt{k}\,.\,\rho$, where k is a non-square number or a fraction m/n which, when reduced to its lowest terms, is not square. In regard, therefore, to rational *straight lines* (only) Euclid takes a somewhat broader view than we have met before. On the other hand, the straight lines $(1 \pm \sqrt{k})\rho$ and $(\sqrt{k} \pm \sqrt{\lambda})\rho$ corresponding to $\sqrt{a} \pm \sqrt{b}$ in algebra (when $\sqrt{a}$, $\sqrt{b}$ are not commensurable) are irrational.

The area $\sqrt{k}\,.\,\rho^2$ which may be regarded as a rectangle with sides ρ and $\sqrt{k}\,.\,\rho$ is a *medial* rectangle or area, and the side of a square equal to it, or $k^{\frac{1}{4}}\rho$, is a *medial straight line*, the first in Euclid's classification of irrational straight lines (it is, of course, the mean proportional between ρ and $\sqrt{k}\,.\,\rho$). The medial straight line may take any equivalent forms, e.g. $\sqrt{(b\sqrt{A})}$ or $\sqrt[4]{(AB)}$.

The Book opens with the famous proposition (X. 1) which is the basis of the method of exhaustion used in Book XII, namely that, if from any magnitude there be subtracted more than its half (or its half), from the remainder again more than its half (or its half), and so on continually, there will at length remain a magnitude less than any assigned magnitude of the same kind. Proposition 2 uses the operation for finding the greatest common measure of two magnitudes as a test of their commensurability or otherwise; Propositions 3, 4 find the greatest common measure, where there is one, of two or three

magnitudes, just as VII. 2, 3 do for numbers. Easy propositions (5–8) lead up to the fundamental theorem of Theaetetus (9). Propositions 17, 18 prove the equivalent of the fact that the roots of the quadratic equation $ax - x^2 = \frac{1}{4}b^2$ are commensurable or incommensurable with a according as $\sqrt{(a^2 - b^2)}$ is commensurable or incommensurable with a. Propositions 19–21 deal with rational and irrational rectangles, and Propositions 23–8 with *medial* rectangles and straight lines. The difference between two *medial* areas, e.g. $\sqrt{k}\,.\,\rho^2$ and $\sqrt{\lambda}\,.\,\rho^2$ cannot be rational (26); this is equivalent to proving, as we do in algebra, that $\sqrt{k} - \sqrt{\lambda}$ cannot be equal to k'. Next Euclid finds (27, 28) medial straight lines commensurable in square only, (1) containing a *rational* rectangle, e.g. $k^{\frac{1}{4}}\rho$ and $k^{\frac{3}{4}}\rho$, and (2) containing a *medial* rectangle, as $k^{\frac{1}{4}}\rho$, $\lambda^{\frac{1}{2}}\rho/k^{\frac{1}{4}}$.

Two lemmas follow, the object of which is to find (1) two square numbers the sum of which is a square number, (2) two square numbers the sum of which is not square. Euclid's solution in the first case has been given above (p. 48); the numbers found in the second case are $mp^2\,.\,mq^2$ and $\{\frac{1}{2}(mp^2 - mq^2) - 1\}^2$.

Propositions 29, 30 find two *rational* straight lines x, y commensurable in square only such that $\sqrt{(x^2 - y^2)}$ is (1) commensurable, (2) incommensurable, with x, and Propositions 31, 32 four pairs of *medial* straight lines x, y commensurable in square only satisfying the four possible combinations of the conditions of xy being rational or medial and $\sqrt{(x^2 - y^2)}$ commensurable or incommensurable with x. Euclid then finds (33–5) *three* pairs of lines x, y *incommensurable in square* satisfying the respective sets of conditions (1) $x^2 + y^2$ rational, xy medial, (2) $x^2 + y^2$ medial, xy rational, (3) $x^2 + y^2$, xy both medial and incommensurable with one another.

With Proposition 36 begins Euclid's exposition of compound irrational straight lines, each of which is the sum or difference of two straight lines incommensurable in length. The first set contains six with the positive sign (Props. 36–41) and six with the negative sign (Props. 73–8). The first pair is the sum and difference of two rational straight lines commensurable in square only, e.g. $\rho \pm \surd k \,.\, \rho$ (where, of course, ρ may be of the form a or $\surd A$). The second, third, fourth, fifth, and sixth pairs are the sums and differences of the pairs of lines x, y found in Props 27, 28, 33, 34, 35 respectively. The names of the first pair are *binomial* and *apotome* respectively; those of the other five pairs are more complicated. As a matter of fact, these six pairs of compound irrationals are the positive roots of different equations of the form

$$x^4 \pm 2\alpha x^2 \,.\, \rho^2 \pm \beta\rho^4 = 0,$$

where ρ is a rational straight line and α, β have different characters and value (α, but not β, may contain a surd, as $\surd m$ or $\surd(m/n)$, as well as rational numbers).

Take the equation $x^4 - 2\alpha x^2 \,.\, \rho^2 + \beta\rho^4 = 0$; then, solving for x^2, we have

$$x^2 = \rho^2\{\alpha \pm \surd(\alpha^2 - \beta)\}.$$

Now x is a compound irrational which has to be expressed as the sum or difference of two terms. Therefore we have to express $\surd\{\alpha \pm \surd(\alpha^2-\beta)\}$ as the sum or difference of two terms. We should find these terms (u, v, say) thus.

Suppose that $\quad u^2+v^2 = \alpha,$

and $\quad 2uv = \surd(\alpha^2-\beta), \quad$ or $\quad 4u^2v^2 = \alpha^2-\beta.$

By subtraction,

$$(u^2-v^2)^2 = \beta, \quad \text{or} \quad u^2-v^2 = \surd\beta.$$

Therefore $\quad u^2 = \tfrac{1}{2}(\alpha+\surd\beta)$, and $v^2 = \tfrac{1}{2}(\alpha-\surd\beta)$,

and the required compound irrational straight lines are

$$\surd\{\tfrac{1}{2}(\alpha+\surd\beta)\} \pm \surd\{\tfrac{1}{2}(\alpha-\surd\beta)\}.$$

Euclid does the exact geometrical equivalent of this working in Propositions 54–9 and 91–6.

Propositions 42–7 and 79–84 prove that each of the twelve compound irrational straight lines forming the first set is divisible into its terms in only one way. In particular, Proposition 42 is equivalent to the well-known theorem in algebra that,

if $\quad a+\surd b = x+\surd y, \quad$ then $\quad a=x,\ b=y,$

and,

if $\surd a+\surd b = \surd x+\surd y$, then $a=x$, $b=y$, or $a=y$, $b=x$.

In Propositions 48–53 and 85–90 Euclid sets out the second set of six pairs of compound irrationals which are called the *first*, *second*, *third*, *fourth*, *fifth* and *sixth binomials*, and the *first*, *second*, *third*, *fourth*, *fifth* and *sixth apotomes* respectively, according as the terms are connected by the positive or negative sign. These irrationals are the positive roots of quadratic equations of the form

$$x^2 \pm 2\alpha x \,.\, \rho \pm \beta\rho^2 = 0,$$

where α, β have different values and character, as before.

Take the equation $x^2 - 2\alpha x \,.\, \rho + \beta\rho^2 = 0$; this gives

$$x = \rho\{\alpha \pm \surd(\alpha^2-\beta)\}.$$

It remains to prove the reciprocal connexion between the two sets of compound irrationals in pairs. We should express it by saying that one of the second set is the *square* of its analogue in the first set, and that one of the first set is the *square root* of its analogue in the second set. Euclid states the facts in a geometrical form, but his geometrical proofs correspond to what we should do in algebra

(Propositions 54–65 and 91–102). For example, Proposition 54 proves that the side of a square equal to the rectangle contained by ρ and the '*first binomial*' is a '*binomial*', and Proposition 60 proves that the square on a '*binomial*' if applied to a rational straight line (σ, say) has for its breadth a '*first binomial*', and so on.

Straight lines commensurable in length with any of the twelve compound irrationals are irrationals of the same type and order respectively (66–70 and 103–7). Finally, it is proved at the end of Proposition 72 and in Proposition 111 and the explanation following it that the medial and the twelve other irrationals are all different from one another.

Propositions 112–14 are the equivalent of rationalizing the denominators of the fractions $c^2/(\sqrt{A}\pm\sqrt{B})$ or $c^2/(a\pm\sqrt{B})$ by multiplying numerator and denominator by $(\sqrt{A}\mp\sqrt{B})$ or $(a\mp\sqrt{B})$ respectively.

Fuller details will be found in *The Thirteen Books of Euclid's Elements*, vol. iii.

What, it may be asked, is the specific object of the elaborate classification in Book X? The most probable explanation seems to be this. In algebra we can express any root of an equation such as those mentioned above in symbolic form by means of surds. The Greeks had no such symbols; the roots of the equivalent equations found by geometry are always straight lines, any one of which looks like any other. The Greeks, therefore, seem to have thought it necessary to compile a sort of repertory of results, described in definitions instead of by symbols, so that if, for instance, a certain straight line which has to be found in a particular case is proved to be a particular irrational, a 'binomial' or 'apotome', a 'major' or a 'minor' irrational, and so on, this would be accepted as a sufficient solution; cf. the straight line proved in XIII. 17 to be an 'apotome', that in XIII. 6 proved to be a 'first apotome', and other similar cases occurring in Pappus.

Books XI–XIII are almost entirely concerned with geometry in three dimensions. The definitions are in Book XI, and include those of a straight line, and a plane, at right angles to a plane, the inclination of a plane to a plane (dihedral angle), parallel planes, equal and similar solid figures, solid angle, pyramid, prism, sphere, cone, cylinder, and parts of them, cube, octahedron, icosahedron, and dodecahedron. The sphere is defined, not as having all the points on its surface equidistant from the centre, but as the figure comprehended by the revolution of a semicircle about its diameter; this is clearly with an eye to the propositions in Book XIII where the regular solids have to be 'comprehended' in a sphere respectively.

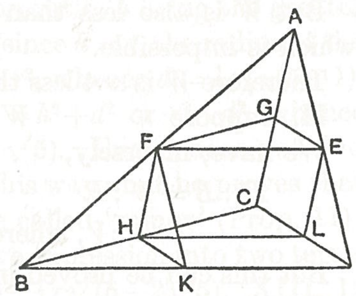
The order of propositions in Book XI is fairly analogous to the order followed in Books I and VI. A straight line is wholly in a plane if a portion of it is in the plane (Prop. 1), and two intersecting straight lines are in one plane, as is a triangle also (2). Straight lines perpendicular to planes are next dealt with (4–6, 8, 11–14), then parallel straight lines not all in the same plane (9, 10, 15), parallel planes (14, 16), planes at right angles to one another (18, 19), solid angles contained by three plane angles (20, 22, 23, 26) or by more plane angles (21). The rest of the Book is mainly on parallelepipedal solids. Thus parallelepipedal solids on the same or equal bases and between the same parallel planes (i.e. having equal heights) are equal (29–31). Parallelepipedal solids of equal height are to one another as their bases (32). Similar parallelepipedal solids are in the triplicate ratio of corresponding sides (33). In equal parallelepipedal solids the bases are reciprocally proportional to the heights, and conversely (34). If four straight lines be proportional, so are similar parallelepipedal solids similarly described upon them, and conversely (37).

In Book XII the *method of exhaustion* plays the leading part, being used to prove successively that circles are to one another as the squares on their diameters (Props. 1, 2), that pyramids of the same height and with triangular bases are to one another as their bases (3–5), that any cone is, in content, equal to one third part of the cylinder which has the same base with it and equal height (10), that cones and cylinders of the same height are to one another as their bases (11), that similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases (12), and finally that spheres are to one another in the triplicate ratio of their diameters (16–18).

Proposition 5 is extended to pyramids with polygonal bases by Proposition 6; and Proposition 7 proves that any prism with triangular bases is divided into three pyramids with triangular bases and equal in content, whence it follows that any pyramid with triangular base (and therefore also any pyramid with polygonal base) is equal to one third part of the prism having the same base and equal height. Lastly, we have propositions about pyramids, cones, and cylinders similar to those in Book XI about parallelepipeds and in Book VI about parallelograms; similar pyramids are in the triplicate ratio of corresponding sides (8), and in equal pyramids, cones, and cylinders the bases are reciprocally proportional to the heights, and conversely (9, 15).

The method of exhaustion, as applied in Euclid, rests of course on X. 1 as lemma. The case of the pyramid (pyramids with triangular bases and of the same height are to one another as their bases) may be given as an illustration.

It is first proved (Proposition 3) that, given any pyramid, as $ABCD$, on the base BCD, if we bisect the six edges at the points E, F, G, H, K, L, and draw the straight lines shown in the figure, we divide the pyramid into two equal prisms and two equal pyramids $AFGE$, $FBHK$ similar to the original pyramid (the equality of the prisms is proved in XI. 39), and that the sum of the two prisms is greater than half the original pyramid.



Proposition 4 proves that, if each of two given pyramids of the same height be so divided, and if the small pyramids in each be similarly divided, then the smaller pyramids left over from that division similarly divided, and so on to any extent, the sums of all the pairs of prisms in the two given pyramids respectively will be to one another as the respective bases. Let the two pyramids and their volumes be denoted by P, P' respectively, and their bases by B, B' respectively. Then, if $B : B'$ is not equal to $P : P'$, it must be equal to $P : W$, where W is some volume either less or greater than P'.

I. Suppose $\qquad W < P'$.

By X. 1 we can divide P' and the successive pyramids in it into prisms and pyramids until the sum of the small pyramids left over in it is less than $P' - W$, so that

$$P' > (\text{prisms in } P') > W.$$

Suppose this done, and P divided similarly.

Then (XII. 4)

$$(\text{sum of prisms in } P) : (\text{sum of prisms in } P') = B : B'$$
$$= P : W, \text{ by hypothesis.}$$

But $\quad P >$ (sum of prisms in P);
therefore $\quad W >$ (sum of prisms in P').

But W is also less than the sum of the prisms in P': which is impossible.

Therefore W is not less than P'.

II. Suppose $\quad W > P'$.

We have, inversely,

$$B' : B = W : P$$
$$= P' : V, \text{ where } V \text{ is some solid less than } P.$$

But this can be proved impossible, exactly as in Part I.

Therefore W is neither greater nor less than P', so that

$$B : B' = P : P'.$$

Book XIII crowns the work by showing how to construct and to 'comprehend in a sphere' each of the five regular solids, the pyramid or tetrahedron (Prop. 13), the octahedron (14), the cube (15), the icosahedron (16), and the dodecahedron (17): 'comprehending in a sphere' means of course finding the sphere which circumscribes each solid, and this involves the determination of the relation of a 'side' (i.e. edge) of the solid to the radius of the sphere: in the case of the first three solids the relation is actually determined, while in the case of the icosahedron the side of the figure is shown to be the irrational straight line called 'minor' (cf. X. 76), and in the case of the dodecahedron an 'apotome'. Preliminary propositions relate to straight lines cut in extreme and mean ratio (1–6) and to pentagons (7, 8), and it is proved that if, in a regular pentagon, two diagonals (straight lines joining angular points next but one to each other) be drawn meeting in a point, each of them is divided at the point in extreme and mean ratio, and the greater segment is equal to the side of the pentagon. Propositions 9, 10 relate to the sides of

a regular pentagon, decagon, and hexagon all inscribed in one circle. If p, d, h be the sides of the figures respectively, $h+d$ is cut in extreme and mean ratio, h being the greater segment (9); this is equivalent (since $h = r$, the radius of the circle) to saying that $(r+d)d = r^2$, whence $d = \frac{1}{2}r(\sqrt{5}-1)$. Proposition 10 proves that $p^2 = h^2+d^2$ or r^2+d^2, whence we can deduce $p = \frac{1}{2}r\sqrt{(10-2\sqrt{5})}$. Euclid does not find p, the side of the pentagon, in this way; but he proves that it is the irrational straight line called 'minor' (Prop. 11); we can in fact separate the above expression into two terms and deduce $p = \frac{1}{2}r\sqrt{(5+2\sqrt{5})} - \frac{1}{2}r\sqrt{(5-2\sqrt{5})}$. XIII. 12 proves that, if a is the side of an equilateral triangle inscribed in a circle with radius r, $a^2 = 3r^2$.

The constructions (only) for the several solids are as follows:

1. The regular pyramid or *tetrahedron.*

Given D, the diameter of the sphere which is to circumscribe the tetrahedron, Euclid draws a circle with radius r such that $r^2 = \frac{1}{3}D \cdot \frac{2}{3}D$, or $r = \frac{1}{3}\sqrt{2} \cdot D$, inscribes an equilateral triangle in the circle, and then erects from the centre of it a straight line perpendicular to its plane and of length $\frac{2}{3}D$. The lines joining the extremity of this perpendicular to the angular points of the equilateral triangle form, with the triangle itself, the required tetrahedron.

2. The *octahedron.*

If D be the diameter of the circumscribing sphere, a square is inscribed in a circle with D as diameter, and from its centre straight lines are drawn in both directions perpendicular to its plane and of length equal to the radius of the circle. Joining the extremities of the perpendiculars to the four angular points of the square, we have the required octahedron.

3. The *cube*.

D being the diameter of the circumscribing sphere, draw a square with side a such that $a^2 = D.\frac{1}{3}D$, and describe a cube on this square as base.

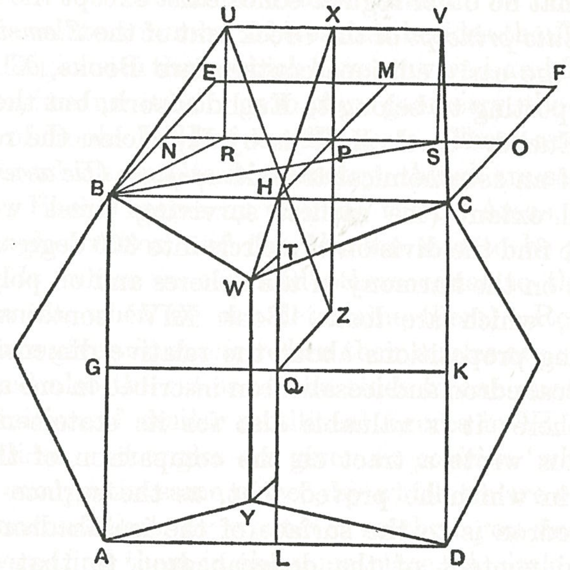
4. The *icosahedron*.

Given D, the diameter of the circumscribing sphere, describe a circle with radius r such that $r^2 = D.\frac{1}{5}D$. Inscribe a regular decagon in the circle. From the angular points draw straight lines at right angles to the plane of the circle (on one side of it) and of length r. This determines the angular points of another regular decagon inscribed in an equal parallel circle. Join alternate angular points in one decagon, making a regular pentagon inscribed in the same circle; do the same in the other circle, but so that the angular points of the second pentagon are not opposite those of the first pentagon. Join the angular points of one pentagon to the nearest angular points of the other; this gives ten equilateral triangles forming part of the surface of the required solid. To find the remaining faces, draw from the centre of each circle (outwards, i.e. in the direction away from the other circle in each case) perpendiculars of such length that the lines joining the extremity of each perpendicular to the five angular points of the nearer of the pentagons are all equal to the side of the pentagon. This gives the ten equilateral triangles which complete the required icosahedron. (The length of each perpendicular is actually equal to the side of the regular decagon inscribed in the circles.)

5. The *dodecahedron*.

Given a sphere with diameter D, Euclid first inscribes in it a *cube*. He then draws regular pentagons which have the sides of the cube for 'diagonals' in the manner shown

in the annexed figure, thus. In one face BF let HM, NO be straight lines joining the middle points of opposite sides



and intersecting at right angles at P; and in the face BD let HL, GK be similarly drawn, meeting in Q. Divide PN, PO, QH in extreme and mean ratio at R, S, T, and let PR, PS, QT be the greater segments. Draw (outwards) RU, PX, SV at right angles to the face BF, and TW at right angles to the face BD, such that each of these perpendiculars is equal to PR or PS. Join UV, VC, CW, WB, BU.

It is then shown that $UVCWB$ is one of the required pentagonal faces by proving that the figure is (1) equilateral, (2) in one and the same plane, and (3) equiangular. The other pentagons are similarly drawn in relation to the other edges of the cube.

Book XIII ends with Proposition 18, which arranges in order of magnitude the edges of the five regular solids inscribed in one and the same sphere; and an addendum proves that no other regular solids exist except the five.

The *editio princeps* of the Greek text of the *Elements* and most of the early editions contain two Books, XIV and XV, purporting to belong to Euclid's work, but these are not by Euclid. 'Book XIV' is by Hypsicles, the reputed author of an astronomical tract *Ἀναφορικός* (*De ascensionibus*) still extant (the earliest surviving Greek work in which we find the division of a circle into 360 degrees), and of works on the harmony of the spheres and on polygonal numbers, which are lost. 'Book XIV' contains some interesting propositions about the relative dimensions of the dodecahedron and icosahedron inscribed in one and the same sphere; it is valuable also for its statement that Apollonius wrote a tract on the comparison of the two figures, in which he proved that, as the surface of the dodecahedron is to the surface of the icosahedron, so is the solid content of the dodecahedron to that of the icosahedron, 'because the perpendicular from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron is the same'. We also learn from Hypsicles that Aristaeus, in a work entitled *Comparison of the five figures*, proved that the same circle circumscribes the pentagon of the dodecahedron and the triangle of the icosahedron inscribed in the same sphere.

'Book XV' is also concerned with the regular solids, but is badly arranged and is of no particular interest, except for the fact that in the third section of it we are given rules for determining the dihedral angles between the faces meeting in any edge of any one of the regular solids, and the rules are attributed to 'Isidorus, our great teacher', who is doubtless Isidorus of Miletus, the architect of the church of St. Sophia at Constantinople (about A.D. 532).

EUCLID'S OTHER WORKS

Euclid wrote a number of treatises besides the *Elements*. We will begin with those which have survived, and first with the *Data*, because it belongs to plane geometry, the subject-matter of Books I–VI of the *Elements*. There are several senses in which, in Greek geometry, things are said to be 'given'; Euclid begins by defining them. Areas, straight lines, angles, ratios, and the like are said to be given *in magnitude* 'when we can find others equal to them' (in other words, when we can determine them). Rectilineal figures are given *in species* when their angles are severally given and also the ratios of the sides to one another (cf. the definition of similar rectilineal figures in VI, Def. 1). Points, lines, and angles are given *in position* 'when they always occupy the same place', by which we are no doubt to understand that, by whatever method you find them, you always find them in the same place. A circle is given *in position and in magnitude* when the centre is given in position and the radius in magnitude.

The object of the type of proposition formulated in the *Data* is to prove that, if in a given figure certain parts or relations are given, other parts or relations are also given, in one or other of the senses defined. It is manifest that a collection of propositions of this form is calculated to shorten the procedure in the analysis preliminary to a problem or proof; this is no doubt the reason why Pappus included the *Data* of Euclid in the *Treasury of Analysis*. Provided that we know that a certain thing is given, it is often unnecessary to carry out the actual operation of determining it.

As we should expect, much of the subject-matter of the *Data* is the same as that of the *Elements*, but in a different