V

PYTHAGOREAN GEOMETRY

About fifty years separated Thales and Pythagoras. With Pythagoras geometry became for the first time a scientific subject pursued for its own sake. 'Pythagoras', says the Proclus-summary, 'transformed the study of geometry into a liberal education, examining the principles of the science from the beginning and probing the theorems through and through in a purely intellectual manner'. Favorinus says that he 'used definitions on account of the mathematical nature of the subject'. We conclude that Pythagoras first laid down certain principles (including definitions), and then built up an ordered sequence of propositions. 'A figure and a platform, not a figure and sixpence'; this was the Pythagorean motto, meaning that each new theorem sets up a platform from which to ascend to the next, and so on.

A comparatively early authority, Callimachus (about 250 B.C.), is quoted by Diodorus as having said that Pythagoras discovered some geometrical theorems himself and was the first to introduce others from Egypt into Greece. Five lines quoted by Diodorus (*minus* a few words) also form part of a longer fragment in the Oxyrhynchus Papyri, though the text is still uncertain. The verses tell us about the cup bequeathed by Bathycles, an Arcadian, to be given to the best of the Seven Wise Men, and how it was first brought to Thales by Bathycles' son, who 'by a happy chance found the old man scraping the ground and drawing the figure discovered by the Phrygian Euphorbus, who was the first to draw even scalene triangles and a circle . . .'. Euphorbus is of course Pythagoras, who claimed to have been Euphorbus in one of his various

incarnations. And, in spite of the anachronism, the figure discovered by Euphorbus is presumably the theorem of the square on the hypotenuse. The rest is uncertain. After the word 'circle' (*κύκλον*) Diodorus has *ἐπταμήκη*, 'seven-lengthed', which, if correct, can hardly be taken to mean anything else but the circle including seven orbits, i.e. the zodiac circle, which embraces the independent circles of the sun, moon, and planets. But this leaves the words 'even scalene triangles' high and dry, as it were. It would be more natural if the reading were such as to enable us to connect the circle with the scalene triangle, e.g. if the circle were the circle circumscribing the scalene triangle.[1] If Thales actually circumscribed a circle about a right-angled triangle, as the citation from Pamphile suggests, it would be most appropriate that the Pythagoreans should generalize the problem and show how to circumscribe a circle about *any* scalene triangle.

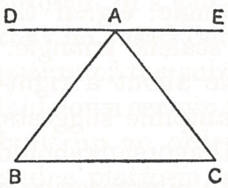
We proceed to set out the propositions in geometry which are definitely attributed to the Pythagoreans, including those associated with the name of Pythagoras himself.

(*a*) *The sum of the angles of any triangle is equal to two right angles.*

As we have seen (p. 88), it is likely enough that this was first discovered with reference to the particular case

[1] Diodorus' reading of the line in question is *καὶ κύκλον ἐπταμήκη δίδαξε νηστεύειν*, which does not scan. The Papyrus, in place of *ἐπταμήκη*, has what looks like ἐ$\overset{\lambda}{\underset{.}{\pi}}$, and the rest of the line apparently blank. Diels reads *καὶ κύκλον ἕ⟨λικα⟩ κἠδίδαξε νηστεύειν*. I should like to suggest *καὶ κύκλον ἔμπλην* instead of *καὶ κύκλον ἕ⟨λικα⟩*, the word *ἔμπλην* meaning 'next to' or 'close by', which seems a possible description (in a poem) of a circle *circumscribed* about a triangle.

of a right-angled triangle, after which the extension of the theorem to any triangle would be made by dividing the triangle, by a perpendicular drawn from a vertex to the opposite side, into two right-angled triangles. All that we are told, however, is that Eudemus attributed the discovery of the general theorem to the Pythagoreans and gave their proof of it. This proof, as elegant as that of Euclid, depends, equally with his, on the properties of parallels, which must therefore have been known to the authors of it. It is as follows:



Let ABC be any triangle, and through A draw DAE parallel to BC.

Then, since BC, DE are parallel, the alternate angles DAB, ABC are equal.

Similarly, the alternate angles EAC, ACB are equal.

Therefore the sum of the angles ABC, ACB is equal to the sum of the angles DAB, EAC.

Add to each sum the angle BAC; therefore the sum of the three angles ABC, ACB, BAC, i.e. the three angles of the triangle, is equal to the sum of the three angles DAB, BAC, CAE, i.e. to two right angles.

We need not hesitate to credit the Pythagoreans with the more general propositions about the angles of any polygon, namely (1) that, if n be the number of the sides or angles, the interior angles of the polygon are together equal to $2n-4$ right angles, and (2) that the exterior angles of the polygon (being the supplements of the interior angles respectively) are together equal to four right angles. The propositions are interdependent, and Aristotle twice quotes the latter. The Pythagoreans also discovered that the only three regular polygons the angles of which, if placed together round a common point as vertex, will just fill up the space (four right angles) round the point are the equilateral triangle, the square, and the regular hexagon.

(β) *The 'Theorem of Pythagoras'* (=Eucl. I. 47).

Tradition is unanimous in referring to Pythagoras the discovery of the theorem of the square on the hypotenuse; but the documentary evidence is far from conclusive. Callimachus speaks of the 'figure' discovered by Pythagoras, and the distich of Apollodorus the 'calculator' or mathematician (date uncertain) says, 'When Pythagoras discovered that famous proposition on the strength of which he offered a splendid sacrifice of oxen'. Unfortunately neither author says what the proposition he refers to actually was. But Plutarch, Athenaeus, Diogenes Laërtius, and Porphyry all connect the story of the sacrifice with the theorem of the square on the hypotenuse, though Plutarch, in giving Apollodorus' verses, expresses doubt whether the proposition referred to is that theorem or a certain problem of 'applying an area', while in another passage he says that the occasion of the sacrifice was the solution of the problem 'given two (rectilineal) figures, to apply [he should rather have said 'construct'] a third which shall be equal to the one and similar to the other'. Vitruvius, however, a century or so before Plutarch, definitely connected the sacrifice with the discovery that the particular triangle 3, 4, 5 is right angled. Proclus will not commit himself to a definite opinion; he says, 'If we listen to those who wish to recount ancient history, we may find some of them referring this theorem (Eucl. I. 47) to Pythagoras and saying that he sacrificed an ox in celebration of his discovery. But, for my part, while I admire *those who* first observed the truth of the theorem, I marvel more at the writer of the Elements, not only because he confirmed it by a most lucid demonstration, but because he compelled assent to the still more general theorem in the sixth book by the irrefutable arguments of science'. It is difficult for us to be more positive than Proclus was; but for myself I like to believe that, so far as the general theorem and the proof of it are concerned, the commonly accepted tradition is right.

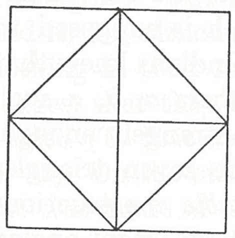
Some knowledge, however, of the property of right-angled triangles can be traced long before the date of Pythagoras. The Egyptians indeed do not seem to have had it, for, although they knew that $3^2+4^2 = 5^2$, there is nothing in their mathematics, so far as known to us, to suggest that they knew that the triangle (3, 4, 5) is right-angled (T. Eric Peet, *The Rhind Mathematical Papyrus*, p. 32). On the other hand, it would appear that practical use was made of the theorem of the square on the hypotenuse, as early as (say) 2000 B.C., by the Babylonians. The evidence for this is the text of certain Babylonian tablets containing mathematical problems which have just recently (1928–9) been interpreted for the first time by O. Neugebauer, W. Struve, and others. Two of the problems are: to calculate the length (1) of a chord of a circle from its *sagitta* and the diameter of the circle, and (2) of the *sagitta* from the chord and the diameter. If c be the chord, a its *sagitta*, and d the diameter of the circle, the formulae intended to be used are evidently $c = \sqrt{\{d^2-(d-2a)^2\}}$ and $a = \frac{1}{2}\{d-\sqrt{(d^2-c^2)}\}$, and it is not possible to account for these formulae except on the assumption that they were based, in some form or other, on the theorem of Pythagoras. In the particular case $a = 2$, $c = 12$, $d = 20$, and the property used is $20^2 = 16^2+12^2$, equivalent to $5^2 = 4^2+3^2$.

Again, there are those who credit the Indians with the discovery of the theorem. The claim is mainly based on the *Āpastamba-Śulba-Sūtra*, which is thought to be at least as early as the fourth or fifth century B.C. A feature in this work is the construction of right angles by means of stretched cords in the ratios of the sides of certain right-angled triangles in rational numbers. Seven such triangles are used, which however reduce themselves to four, namely (3, 4, 5), (5, 12, 13), (8, 15, 17), and (12, 35, 37).

One of these triangles (5, 12, 13) was known as early as the eighth century B.C., while yet another (7, 24, 25) appears in the *Bāudhāyana Ś.-S.*, which is supposed to be earlier than Āpastamba. Hence the Indians knew that five distinct triangles in rational numbers a, b, c such that $a^2+b^2=c^2$ are right angled. Yet, strangely enough, Āpastamba says, with reference to the seven triangles which he mentions, 'so many *recognizable* constructions are there', as if he knew of no other rational right-angled triangles. But Āpastamba does also state the equivalent of Eucl. I. 47 in general terms, though without proof, and bases on it constructions for finding the square equal to (1) the sum, (2) the difference, of two given squares. He also recognizes the truth of the theorem for an isosceles triangle, and even gives a construction for $\sqrt{2}$ or the length of the diagonal of a unit-square; he in fact constructs a line which is $\left(1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}\right)$ times the side. This approximation to $\sqrt{2}$ is no doubt derived from the consideration that $2.12^2 = 17^2-1$, but the author does not betray any knowledge of the fact that this approximate value is not exact.

The Indians, therefore, knew empirically of the property of right-angled triangles and stated it generally. But they gave no indication of any proof; their statement appears to have been the result of an imperfect induction from a very small number of cases of right-angled triangles in rational numbers known to them. This is in great contrast to what is attributed to Pythagoras, which includes the discovery of a general formula for finding an unlimited number of rational right-angled triangles.

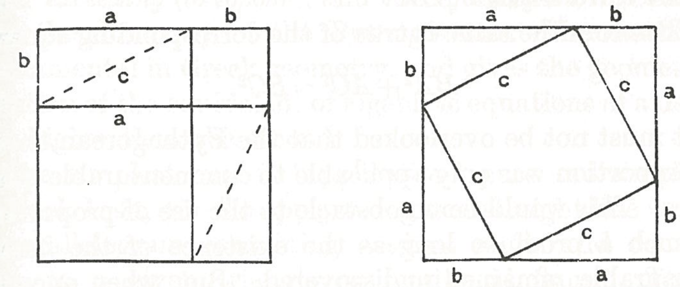
Assuming that, as Vitruvius says, Pythagoras began with the triangle (3, 4, 5), the next step would be to seek for other similar cases. An experiment may have been made with an isosceles right-angled triangle, and the mere drawing of a figure would indicate the property in this case. If the middle points of the sides of a square be joined in order as in the annexed diagram, we have a square inside the original square and obviously half as large. But the original square is equal to the square on the diagonal of the inner square; therefore the square on the diagonal is equal to twice the square on the side.



In suggesting a possible method by which the general theorem was first proved we have a choice between two different lines of proof. One would be to represent the three squares in a figure, and to show how the two are equal to the one; this would be after the manner of Euclid, Book II. The other would be to use proportions after the manner of Euclid, Book VI.

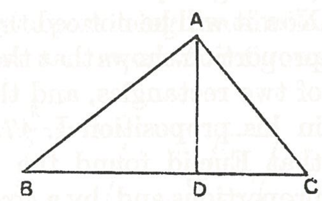
If the first method is preferred, no better suggestion can be made than that of Bretschneider and Hankel. The first of the subjoined figures, which is like that of Euclid, II. 4, represents a larger square of side $(a+b)$ and two smaller squares of sides a, b respectively, with the two complementary rectangles (a, b). Dividing each com-

plementary rectangle into two equal right-angled triangles by drawing the diagonal c, we then dispose the four triangles within another square of side $(a+b)$ as shown in the second figure. Deducting the four right-angled triangles



(a, b, c) from the larger square in each figure, we have as remainders, (1) in the first figure, the squares on a and b, and (2) in the second figure, the one square on c. Therefore the sum of the squares on a, b is equal to the square on c.

The proof by proportion might take different forms. Let ABC be a triangle right angled at A. Draw AD perpendicular to BC. Then the triangles DBA, DAC are similar to the triangle ABC and to one another.



Now (1) it follows from the theorems of Eucl. VI. 4 and 17 that

$$BA^2 = BD \cdot BC,$$

and

$$AC^2 = CD \cdot BC.$$

Therefore

$$BA^2 + AC^2 = BC^2.$$

Alternatively (2) it would be seen that, in the similar triangles DBA, DAC, ABC, the corresponding sides opposite to the right angle in each case are BA, AC, BC.

The triangles are therefore in the duplicate ratio of those sides, and so are the squares on the latter. Therefore the squares are proportional to the corresponding triangles.

But two of the triangles, DBA, DAC, together make up the third triangle ABC.

Therefore the same is true of the corresponding squares, or
$$BA^2+AC^2=BC^2.$$

It must not be overlooked that the Pythagorean theory of proportion was only applicable to commensurable quantities. This would be no obstacle to the use of proportions in such a proof so long as the existence of the incommensurable remained undiscovered. But, when once the incommensurable was discovered, it would be necessary, pending the appearance of a new theory of proportion applicable to incommensurable as well as to commensurable magnitudes, to invent new proofs independent of proportions in place of those in which proportions were used. Now it will be noticed that the first of the above proofs by proportion shows that the square on BC is equal to the sum of two rectangles, and this is *precisely* what Euclid proves in his proposition I. 47. It appears probable therefore that Euclid found the proposition proved by means of proportions and, by a stroke of genius, gave the proof a different form in order to get the proposition into Book I in accordance with his general arrangement of the *Elements*.

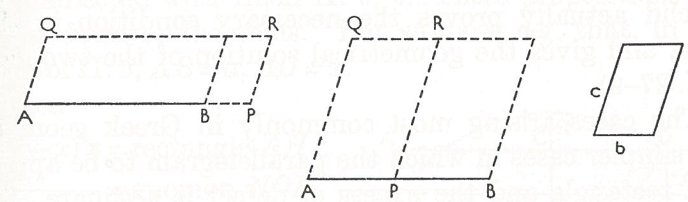
(γ) *Application of areas and geometrical algebra.*

For want of the necessary notation the Greeks had no algebra in our sense. They were obliged to use geometry as a substitute for algebraical operations; and the result is that a large part of their geometry may appropriately be called 'geometrical algebra'. One of the two main

methods at their disposal was the 'application of areas' (the other being the method of proportions). We have it on the authority of Eudemus, cited by Proclus, that the method of 'application of areas' (παραβολὴ τῶν χωρίων), their *exceeding* (ὑπερβολή), and their *falling-short* (ἔλλειψις), was the discovery of the Pythagoreans. The method is fundamental in Greek geometry, and gives the geometrical solution of the equivalent of algebraic equations of a degree not higher than the second.

The simplest case is 'application' pure and simple, as in Eucl. I. 44, 45: To apply to a given straight line as base a parallelogram containing a given angle and equal in area to a given triangle or rectilineal figure. This is equivalent to the operation of finding x where $ax=bc$, i.e. of dividing the product bc by a.

The general case where the applied area 'exceeds' or 'falls short' is enunciated thus: To apply to a given straight line a parallelogram equal to a given rectilineal figure and (1) *exceeding* or (2) *falling-short* by a parallelogram similar to a given parallelogram. In the accompanying figures



the parallelograms AR are *applied* to the straight line AB, but in the first figure the base AP overlaps AB and the parallelogram *exceeds* [the parallelogram on AB itself] by the parallelogram BR, while in the second figure the base AP falls short of AB and the parallelogram AR *falls short* by the parallelogram BR. The problem is, given AB, to draw the figure so that the parallelogram AR shall be equal to a given area (C, say), while the excess or defect BR is similar to a given parallelogram. What has in effect to be done is to determine the *size* of the excess or defect BR (its *shape* is determined by the given figure to which it has to be similar); that is to say, to determine one of the sides BP, PR in such a way that, when the figure is completed, the parallelogram AR may be equal to C. Let the ratio of BP to PR be that of b to c, and suppose that $BP=x$. Let $AB=a$. Then $AP=a\pm x$ and $PR=\frac{c}{b}x$. Now the area of the required parallelogram is $m.AP.PR$, that is, $m(a\pm x)\frac{c}{b}x$, where m is a certain constant depending on the size of the given angle BPR (actually the sine of that angle). Hence the equation to be solved is

$$m(a\pm x)\frac{c}{b}x=C.$$

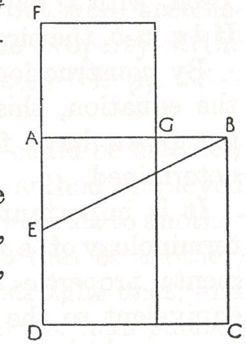
In the case of defect, corresponding to the negative sign, the possibility of a solution is subject to a certain condition. Euclid actually proves the necessary condition in that case, and gives the geometrical solution of the two cases (VI. 27–9).

The cases arising most commonly in Greek geometry are simpler cases in which the parallelogram to be applied is a rectangle and the excess or defect is a square. The corresponding equation is then of the form

$$(a\pm x)\,x=b^2.$$

To solve this equation we should first, if necessary, change the sign throughout so as to make the term in x^2 positive, then add $\frac{1}{4}a^2$ on both sides so as to make the left side a complete square. We have then on the right hand

$\frac{1}{4}a^2\pm b^2$, and we equate the square root of this to the side of the complete square on the left hand. The Greek geometrical procedure was the exact equivalent, as we see from the particular case solved by Euclid in II. 11. We have to divide AB at G so that $AB.BG=AG^2$. If $AB=a$, $AG=x$, this is equivalent to



$$a(a-x)=x^2$$

or

$$x^2+ax=a^2.$$

Euclid bisects AD, the side of the square on AB, at E, and joins EB. Then, after producing EA to F so that $EF=EB$, he makes AG equal to AF.

Now $EB^2=\frac{1}{4}a^2+a^2$
$=(x+\frac{1}{2}a)^2$, from above.

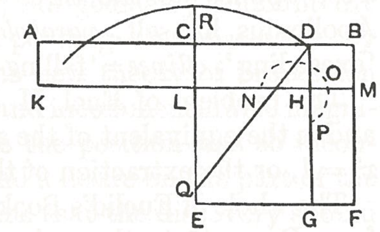
And $EF=EB$; therefore $EF=x+\frac{1}{2}a$, so that $AF=x$, which is therefore found.

The solutions of the cases

$$(a\pm x)\,x=b^2$$

are connected with Eucl. II. 5, 6. These propositions are in the form of theorems. But suppose e.g. that, in the figure of II. 5, $AB=a$, $BD=x$.

Then



$(a-x)\,x=$ rectangle AH
$=$ gnomon NOP.

If, then, the area of the gnomon ($=b^2$, say) is given, we have the equation

$$ax-x^2=b^2,$$

or

$$x^2-ax=-b^2.$$

To solve this equation, we add $\frac{1}{4}a^2$ to both sides and equate $(\frac{1}{2}a-x)$ to the square root of $(\frac{1}{4}a^2-b^2)$. (For a real solution, therefore, b^2 must not exceed $\frac{1}{4}a^2$.)

The geometrical equivalent is this. Bisect AB at C and draw CQ at right angles to AB and of length equal to b. Then, with Q as centre and $\frac{1}{2}a$ as radius, draw a circle. If $\frac{1}{2}a > b$, the circle will cut CB in some point D.

By construction, $CD^2 = QD^2-QC^2 = \frac{1}{4}a^2-b^2$; and, by the equation, this is equal to $(\frac{1}{2}a-x)^2$. Thus by finding D we have found CD or $\frac{1}{2}a-x$, and x, or DB, is determined.

It is important to note that Apollonius employs the terminology of 'application of areas' to describe the fundamental properties of the three conics. These properties are equivalent to the following Cartesian equations referred to axes which are in general oblique:

$$y^2=px \quad \text{(the parabola)},$$

$$y^2=px+\frac{p}{d}x^2 \quad \text{(the hyperbola)},$$

$$y^2=px-\frac{p}{d}x^2 \quad \text{(the ellipse)},$$

where d is the diameter of reference and p the corresponding 'parameter'. This is the origin of the names which were applied to the three conics for the first time by Apollonius himself: *parabola* = 'application', *hyperbola* = 'exceeding', *ellipse* = 'falling-short'.

The problem of Eucl. II. 14 is dependent on I. 44, 45, and is the equivalent of the solution of the pure quadratic $x^2=A$, or the extraction of the square root.

The whole of Euclid's Book II, with the section of Book I from Prop. 42 to the end, may be said to deal with the transformation of areas (or the sums or differences of areas) of rectilineal figures into equivalent areas of different

shape or composition by means of 'application' and the use of the theorem of I. 47. A characteristic of Book II is the use of the *gnomon*, which is essentially Pythagorean. Pythagorean, too, are the theorems of II. 9, 10, which are not only very useful in geometry, but were specifically used for the purpose of proving the property of the successive 'side'- and 'diameter'-numbers (cf. pp. 55–7 above).

The quantitative comparison of areas could be made by means of proportions, the other main method employed in the geometrical algebra. The ratio of one area to another (or of the content of one solid figure to that of another) could be expressed as a ratio between straight lines, and such ratios could be compounded or otherwise manipulated to any desired extent.

(δ) *The irrational.*

The discovery of the incommensurable by the Pythagoreans was bound to cause a great sensation, the more so as it would immediately be seen to throw doubt on so much of the Pythagorean proofs of theorems in geometry as rested on their (arithmetical) theory of proportion. To avoid the *impasse*, it would be necessary to seek proofs on other lines where possible; but geometry undoubtedly suffered a serious set-back pending the discovery by Eudoxus (408–355 B.C.) of the new theory of proportion applicable to commensurable and incommensurable magnitudes alike. In the meantime the position was so inconvenient that we can understand a desire on the part of the inner circle of the Pythagoreans that the discovery should not become known to the profane. This may, perhaps, account for the legend that the first of the Pythagoreans (whether it was Hippasus or another) who made it public perished at sea for his impiety, or (according to another version) was banished from the community and had a tomb erected for him as if he were dead.

(ϵ) *The five regular solids.*

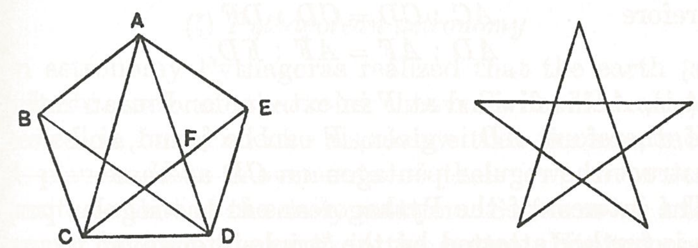
Proclus, speaking of Pythagoras, says in parenthesis that, in addition to a theory of proportionals, he discovered 'the putting-together of the cosmic figures', i.e. the five regular solids. Next we have the story that Hippasus 'was a Pythagorean, but, owing to his having been the first to publish the (construction of the) sphere from the twelve pentagons (i.e. the inscribing of the dodecahedron in a sphere) perished by shipwreck for his impiety, but received credit for the discovery although it really belonged to HIM, for it is thus that they refer to Pythagoras, and they do not call him by his name'.

Now in what sense, if at all, did Pythagoras or the Pythagoreans discover the 'putting-together' of the five regular solids? Some light is thrown on this question by the procedure of Plato in the *Timaeus*. Plato there shows how to construct the regular solids in the elementary sense of putting triangles and pentagons together to form their faces. He forms a square from four isosceles right-angled triangles, and an equilateral triangle from three pairs of triangles which are the halves of equal equilateral triangles cut into two by bisecting one of the angles. Then he forms solid angles by putting together (1) squares three by three, (2) equilateral triangles three by three, four by four, and five by five respectively; the first figure so formed is a cube with eight solid angles, the next three are the tetrahedron, octahedron, and icosahedron respectively. The fifth figure, the dodecahedron, has pentagonal faces, and Plato forms solid angles by putting together equal

pentagons three by three; the result is a regular solid with twelve faces and twenty solid angles.

There is nothing in all this that would be beyond Pythagoras or the Pythagoreans, provided that the construction of the regular pentagon was known to them; the method of formation of the solids agrees well with the known fact that the Pythagoreans put angles of certain regular figures together round a point and showed that only three of such angles would fill up the space in one plane round the point. Moreover, there is evidence of the existence of dodecahedra in very early times. Thus a regular dodecahedron of Etruscan origin discovered on Monte Loffa (Colli Euganei, near Padua) in 1885 is held to date from the first half of the first millennium B.C. It is possible, therefore, that Pythagoras or the Pythagoreans had actually seen such a dodecahedron.

As regards the regular pentagon, we may observe that the construction of it in Euclid, Book IV, depends on the construction of a certain isosceles triangle, which again depends on the problem of cutting a straight line 'in



extreme and mean ratio'. The last-named problem is certainly Pythagorean, being a simple case of 'application of areas'. The construction of the regular pentagon was, therefore, well within the powers of the Pythagoreans. It may have been evolved in some such way as this. Suppose the pentagon constructed, as $ABCDE$. Join AC, AD, CE. Now the Pythagoreans knew the theorems about the sum of the angles of any triangle and the sum of the interior angles of any polygon. They would, therefore, see that each angle, as ABC, of the regular pentagon is $\frac{6}{5}$ths of a right angle. It follows that, in the isosceles triangle BAC, the angle BAC is $\frac{2}{5}$ths of a right angle. So is the angle EAD, and so is the angle ECD, for the like reason. Subtracting the two angles BAC, DAE from the angle BAE (which is $\frac{6}{5}$ths of a right angle), we see that the angle CAD is $\frac{2}{5}$ths of a right angle. So is the angle ACE, for the like reason. It follows that, in the isosceles triangle ACD, each of the base angles is double of the vertical angle.

Again, if AD, CE meet in F, CFD is an isosceles triangle because the angle CFD, being equal to the sum of the angles CAF, ACF, is $\frac{4}{5}$ths of a right angle, and is therefore equal to the angle CDF. Hence $CD = CF = AF$.

Moreover, the triangles ACD, CDF are equiangular and therefore similar;

therefore $\quad AC : CD = CD : DF$

or $\quad AD : AF = AF : FD,$

that is, AD is divided at F in extreme and mean ratio.

If, therefore, AD is given, F can be found, and we can construct the regular pentagon on CD as base.

The interest of the Pythagoreans in the regular pentagon is further attested by the 'triple interwoven triangle' or pentagram, i.e. the star-pentagon which, according to Lucian and the scholiast to the *Clouds* of Aristophanes, was used by the Pythagoreans as a symbol of recognition between members of the same school and was called by them Health. I have drawn the star-pentagon separately,

and the close connexion between the two figures could hardly be missed.

That the dodecahedron is inscribable in a sphere would probably be inferred from a consideration of its regular formation, without constructing the sphere and finding the centre of it in the scientific manner of Eucl. XIII. 17, where the relation between an edge of the dodecahedron and the radius of the sphere is also found: an investigation probably due to Theaetetus. For, according to Suidas, Theaetetus was the first to 'write upon' (or 'construct') the five regular solids, which probably means that he was the first to construct them theoretically and to investigate their relations with the circumscribing spheres and with one another. The scholium No. 1 to Euclid's Book XIII says of that Book that it is about 'the five so-called Platonic figures, which however do not belong to Plato, three of the five being due to the Pythagoreans, namely the cube, the pyramid, and the dodecahedron, while the octahedron and the icosahedron are due to Theaetetus'. This may well have been the case.

(ζ) *Pythagorean astronomy*

In astronomy Pythagoras realized that the earth (and no doubt each of the other heavenly bodies also) is spherical in form, and he also knew that the sun, moon, and planets have movements of their own in a sense opposite to that of the daily rotation. So far as we know, however, he kept the earth in the centre. His successors in the school (one Hicetas of Syracuse and Philolaus are alternatively credited with this innovation) deposed the earth from its place in the centre and made it revolve, like the sun, the moon, and the planets, and an assumed additional body, the 'counter-earth', round the 'central fire', wherein was located the governing principle which directs the movement and activity of the universe. The counter-earth, which accompanies the earth and revolves in a smaller orbit, is not seen by us because the hemisphere of the earth on which we live is turned away from the counter-earth. As the Pythagoreans, according to Aristotle, considered that eclipses of the moon occur owing to the interposition, between it and the sun, sometimes of the earth, sometimes of the counter-earth, the latter may have been invented in order to account for the frequency of lunar eclipses as compared with solar.

SUMMARY.

It may be useful to summarize the contribution of Pythagoras and the Pythagoreans to geometry. With them geometry became a scientific subject studied for its own sake; indeed for Pythagoras geometry was science itself ('geometry was called by Pythagoras "inquiry", ἱστορία'). Pythagoras explored the first principles, starting with definitions, and built upon them a logically connected system.

The positive achievements of the Pythagoreans in geometry were these:

1. They knew the properties of parallels, and used them to prove generally that the sum of the angles of any triangle is equal to two right angles. They deduced the well-known theorems about the sums of (1) the exterior, (2) the interior, angles of any polygon.

2. The transformation of areas of rectilineal figures, and the sums and differences of such areas, into equivalent areas of different shapes, in the manner of Eucl. I. 42–7 and Eucl. II, was their creation. To this end they invented the powerful method of 'application of areas', the main

constituent of the geometrical algebra by which they effected the geometrical equivalent of addition, subtraction, division, extraction of the square root, and finally (with the help of the theorem of the square on the hypotenuse and proportions) the complete solution of the general quadratic equation $x^2 \pm pq \pm q = 0$ so far as it has real roots.

3. They had a theory of proportion pretty fully developed, though it was only applicable to commensurable magnitudes, being presumably a numerical theory on the lines of Euclid, Book VII. They were aware of the properties of similar figures; for Plutarch attributes to Pythagoras himself the solution of the problem of describing a rectilineal figure similar to one given figure and equal in area to another, and this implies a knowledge of the theorem that similar rectilineal figures are in the duplicate ratio of corresponding sides. Much of the content of Euclid, Book VI, must, therefore, have been known to the Pythagoreans.

4. They had discovered, or were aware of the existence of, at least three of the regular solids. There is no reason to doubt that they could construct a regular pentagon in the manner of Eucl. IV. 10, 11.

5. They discovered the existence of the incommensurable in at least one case, that of the diagonal of a square in relation to its side; and they also devised a method of obtaining closer and closer approximations to the value of $\sqrt{2}$ in the form of numerical fractions, by means of the series of 'side'- and 'diameter'-numbers which are the successive solutions of the equations

$$2x^2 - y^2 = \pm 1,$$

for which purpose they used the theorems of Eucl. II. 9, 10.