III

PYTHAGOREAN ARITHMETIC

PYTHAGORAS was born at Samos about 572 B.C. In 532, or thereabouts, in order to escape the rule of Polycrates, tyrant of Samos, he migrated to southern Italy and founded a society at Croton. Then, when the brotherhood, having involved itself in politics, became the subject of plots and attacks, notably by one Cylon, Pythagoras moved to Metapontium, where he died at a great age (some say 75, others 80).

The early philosophers who were nearly contemporary with Pythagoras recognize his wide learning. Heraclitus, who was hostile to him, admits that he, most of all men, cultivated 'inquiry'. Empedocles was his enthusiastic admirer, calling him a man of prodigious knowledge and the greatest master of skilled arts of every kind. Later, Herodotus speaks of him as 'the most able philosopher among the Greeks'.

The first evidence of his work in mathematics is apparently the statement of Aristotle, in his separate book *On the Pythagoreans*, of which only fragments survive, that 'Pythagoras, the son of Mnesarchus, first worked at mathematics and arithmetic, and afterwards, at one time, condescended to the wonder-working practised by Pherecydes'. Elsewhere Aristotle says (*Metaph. A.* 5) that, 'in the time of Leucippus and Democritus and before them, the so-called Pythagoreans applied themselves to the study of mathematics and were the first to advance that science; insomuch that, having been brought up in it, they thought that its principles must be the principles of all existing things'.

The extent of Pythagoras' own share in the mathematical discoveries of his school will always remain uncertain, for he left no written treatises (so it is said), nor apparently was any account of the Pythagorean doctrines written by any one before Philolaus. Aristotle himself evidently felt the difficulty, since he never attributes any philosophical doctrines to the founder of the school but always refers to 'the Pythagoreans' or 'the so-called Pythagoreans'.

Nevertheless, there is good reason to believe the testimony of Proclus that it was Pythagoras who 'transformed the study of geometry into a liberal education, examining the principles of the science from the beginning'; and we can accept the statement of Aristoxenus that Pythagoras attached supreme importance to the study of arithmetic, 'which he advanced and took out of the region of commercial utility'.

We may take it as certain that Pythagoras himself discovered that musical harmonies depend on numerical ratios, the octave representing the ratio 2 : 1 in length of string at the same tension, the fifth 3 : 2 and the fourth 4 : 3. This capital discovery must have created a deep impression; it would confirm in a striking manner (if it was not what actually suggested) the theory that 'all things are numbers'. This theory appears in different forms. Philolaus said that all things *have* numbers, and that it is this fact which enables them to be known. The earliest form of the theory, however, maintained that things *are* numbers. Aristotle, in one place, says of the Pythagoreans that 'they constructed the whole heaven out of numbers, but not of *monadic* numbers, since they suppose the numbers to have *magnitude*'. It is tempting to suppose that this idea arose from observation of the heavens,

which would show that each constellation consists of a certain *number* of stars forming a certain geometrical figure. This would agree with what we learn from Aristotle, who has the interesting remark that one Eurytus, an ancient Pythagorean, 'settled what is the number of what object (e.g. this is the number of a man, that of a horse), and imitated the shapes of living things by pebbles after the manner of those who bring numbers into the forms of triangle or square' (what we call 'figured' numbers).

DEFINITIONS OF THE UNIT AND OF NUMBER

The Pythagoreans connected the unit in arithmetic and the point in geometry by saying that the unit is a 'point without position' (στιγμὴ ἄθετος), and a point is 'a unit having position' (μονὰς θέσιν ἔχουσα). Aristotle observes that the One is reasonably regarded as not itself being a number, since the measure is not measures, but the measure or the One is the beginning (or principle) of number. According to Iamblichus, Thymaridas (an ancient Pythagorean) defined the unit as 'limiting quantity' (περαίνουσα ποσότης) or, as we might say, 'limit of fewness', while some Pythagoreans called it 'the confine between number and parts', i.e. that which separates multiples from submultiples.

The first definition of number is attributed to Thales, who defined it as a 'collection of units' (μονάδων σύστημα), a definition almost identical with Euclid's, namely 'the multitude made up of units'. The Pythagoreans similarly 'made number out of one'; some of them called it 'a progression of multitude beginning from a unit and a regression ending in it'. Eudoxus defined a number as a 'determinate multitude' (πλῆθος ὡρισμένον).

CLASSIFICATION OF NUMBERS

The distinction between *odd* and *even* numbers no doubt goes back to Pythagoras. A Philolaus fragment says that 'number is of two special kinds, odd and even, with a third, even-odd, arising from a mixture of the two; and of each kind there are many forms'. Nicomachus gives the following as the Pythagorean definition of odd and even. 'An *even* number is that which admits of being divided by one and the same operation into the greatest and the least parts, greatest in size but least in number (i.e. into *two* halves) . . . while an *odd* number is that which cannot be so divided, but is only divisible into two unequal parts.' Another ancient definition given by Nicomachus says that 'an *even* number is that which can be divided both into two equal parts and into two unequal parts (except the fundamental dyad, which can only be divided into two equal parts), but, however it is divided, must have its two parts *of the same kind* without share in the other kind (i.e. the two parts are either both odd or both even); while an *odd* number is that which, however divided, must in any case fall into two unequal parts, and those parts always belonging to the two *different* kinds respectively (i.e. one being odd and the other even)'.

How far the early Pythagoreans carried the subdivision of Philolaus' three kinds of number, the odd, the even, and the combination of the two, even-odd, we do not know. But they are not likely to have advanced beyond the point of view of Plato and Euclid, which is much the same. Plato has the terms 'even-times even' (ἄρτια ἀρτιάκις), 'odd-times odd' (περιττὰ περιττάκις), 'odd-times even' (ἄρτια περιττάκις), and 'even-times odd' (περιττὰ ἀρτιάκις), by

which he simply means the product of even and even, odd and odd, odd and even, and even and odd factors respectively. Similarly, according to Euclid, an 'even-times even' number is 'a number measured by an even number according to an even number' (i.e. an even number of times), and an 'even-times odd' number 'a number measured by an even number according to an odd number'. Euclid does not seem, any more than Plato, to have distinguished between an 'even-times odd' number and an 'odd-times even' number, or to have troubled about the fact that, as defined, the classes 'even-times even' and 'even-times odd' are not mutually exclusive; for 24, which is 6 times 4 or 4 times 6, is also 8 times 3.

The Neo-Pythagoreans met this point by a new classification. With them the 'even-times even' number is that which has its half even, the half of the half even, and so on until unity is reached; in short, it is a number of the form 2^n. The 'even-odd' number (ἀρτιοπέρισσος, in one word) is the other extreme, a number which can only be halved once, and then leaves an odd number as quotient, i.e. a number of the form $2(2n+1)$. Intermediate is the third class, the 'odd-even' (περισσάρτιος), a number which can be halved twice or more times successively, but then leaves an odd number as quotient, i.e. a number of the form $2^{n+1}(2m+1)$.

Prime or *incomposite* numbers (πρῶτος καὶ ἀσύνθετος) and *secondary* or *composite* numbers (δεύτερος καὶ σύνθετος) are distinguished in a fragment of Speusippus based upon works of Philolaus. Thymaridas the Pythagorean called a *prime* number *rectilinear*, since it can only be set out in one dimension (the only measure of it, except the number itself, being 1). Alternative terms were *euthymetric* and *linear*. Strictly speaking, the prime number should have

been defined as that which is rectilinear *only*. With Euclid a prime number is 'that which is measured by a unit alone', a composite number 'that which is measured by some number'; numbers *prime to one another* are numbers 'measured by a unit alone as common measure', and numbers *composite to one another* are those 'measured by some number as common measure'. Euclid, as well as Aristotle before him, admitted 2 as a prime number; the Pythagoreans not only excluded 2 from the category of primes, but for them 2, the dyad, was not a number at all, but only the principle of the even, as the unit was the principle of number.

'PERFECT' AND 'FRIENDLY' NUMBERS

'Perfect' numbers in the sense of Euclid's definition (Eucl. VII, Def. 22) appear there for the first time. A 'perfect' number is a number which is equal to (the sum of) all its own parts (i.e. all its factors including 1), e.g.

$$6 = 1+2+3,$$
$$28 = 1+2+4+7+14,$$
$$496 = 1+2+4+8+16+31+62+124+248.$$

Nicomachus knew of these perfect numbers and one other, 8128, and he remarks that they are found in 'ordered' fashion, there being one among the tens, one among the hundreds, and one among the thousands, and that they terminate alternatively in 6 and 8. They do all terminate in 6 or 8 (as can easily be proved by means of the general expression for them established by Euclid, namely $(2^n-1)2^{n-1}$, where 2^n-1 is a prime number), but not alternately, for the fifth and sixth perfect numbers terminate in 6, and the seventh and eighth in 8; and there is no such 'order' in their succession as Nicomachus

supposes. The perfect numbers so far discovered, in addition to the four known to Nicomachus, are:

fifth, $2^{12}(2^{13}-1) = 33\ 550\ 336$
sixth, $2^{16}(2^{17}-1) = 8\ 589\ 869\ 056$
seventh, $2^{18}(2^{19}-1) = 137\ 438\ 691\ 328$
eighth, $2^{30}(2^{31}-1) = 2\ 305\ 843\ 008\ 139\ 952\ 128$
ninth, $2^{60}(2^{61}-1) = 2\ 658\ 455\ 991\ 569\ 831\ 744\ 654\ 692\ 615\ 953\ 842\ 176$
tenth, $2^{88}(2^{89}-1)$
eleventh, $2^{106}(2^{107}-1)$
twelfth, $2^{126}(2^{127}-1)$.

The first five perfect numbers were known in the fifteenth century; the first eight were calculated by Jean Prestet (d. 1670), Fermat (1601–65) having stated that $2^{31}-1$ is prime, a fact proved later by Euler. The remaining four were discovered in the period from 1886 to the present date.

Two numbers are 'friendly' when each is the sum of all the aliquot parts of the other, e.g. 284 and 220 (for $284 = 1+2+4+5+10+11+20+22+44+55+110$, while $220 = 1+2+4+71+142$. Iamblichus refers the discovery of such numbers to Pythagoras, since, the parts of either number producing the other, the numbers realize Pythagoras' definition of a friend as 'Alter ego'. The subject of 'friendly' numbers was taken up by Euler, who discovered no less than sixty-one pairs of them, after Descartes and van Schooten had found only three.

While with Euclid and later writers the 'perfect' number was the kind of number described above, the Pythagoreans (we are told) made 10 the perfect number. Being the sum of the first four numbers 1, 2, 3, 4, the number 10 formed the 'set of four (numbers)' called *tetractys*. This set of numbers includes the numbers out of which are

formed the ratios corresponding to the musical intervals, namely 4 : 3 (the fourth), 3 : 2 (the fifth), and 2 : 1 (the octave). Such virtue was attached to the τετρακτύς that it was for the Pythagoreans their 'greatest oath' and was alternatively called 'Health'. It also gives, when graphically represented by points in four lines one below the other,

a triangular number. Hence Lucian's story that Pythagoras once told some one to count, and, when he had said 1, 2, 3, 4, Pythagoras interrupted 'Do you see? What you take for 4 is 10, a perfect triangle, and our oath'.

Speusippus observes further that 10 contains in it the 'linear', 'plane', and 'solid' varieties of number, for 1 is a point, 2 a line, 3 a triangle, and 4 a pyramid. This is easily seen by placing the right number of dots in the proper positions. We are thus brought to the theory of 'figured' numbers, which seems to go back to Pythagoras himself. It is clear that the oldest Pythagoreans were acquainted with the formation of triangular and square numbers by means of dots or pebbles; and we judge from the account of Speusippus' book *On the Pythagorean Numbers*, which was founded on works of Philolaus, that the latter dealt with linear numbers, polygonal numbers, and plane and solid numbers of all sorts.

FIGURED NUMBERS

(a) *Triangular Numbers.*

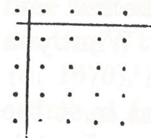
It was probably Pythagoras who discovered that the sum of any number of successive terms of the series of

natural numbers is a triangular number. This is seen diagrammatically thus:

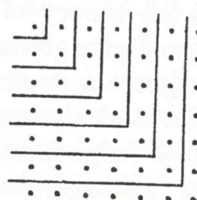
In general, $1+2+3+\ldots+n = \frac{1}{2}n(n+1)$ is a triangular number with side n.

(b) Square numbers and gnomons.

It is easy to see that, if we have a number of dots forming and filling up a square (say 16, as in the accompanying figure), the next higher square, the square of 5, can be formed by adding rows of dots round two sides of the original square as shown. The number of dots so added is $2.4+1 = 9$. This process of forming successive squares can be applied throughout, beginning from the first square number 1. The successive numbers added to the 1 are, as shown in the annexed figure,

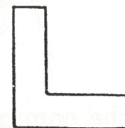


$$3, 5, 7 \ldots (2n+1),$$

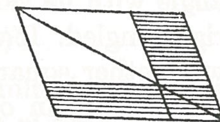


that is to say, the successive odd numbers. The method of formation shows that the sum of any number of successive terms of the series of odd numbers 1, 3, 5, 7 . . . (starting from 1) is a square, and in fact $1+3+5+\ldots+(2n-1) = n^2$, while the addition of the next odd number $2n+1$ makes the next higher square, $(n+1)^2$. All this was known to Pythagoras. The odd numbers successively added were called *gnomons*. Aristotle speaks of gnomons placed round 1 which now produce different figures every time (oblong figures, each dissimilar to the preceding one), now preserve one and the same figure (squares); the latter is the case with the gnomons now in question.

The history of the term *gnomon* is interesting. The gnomon was originally the upright stick which cast shadows on the plane or hemispherical surface of a sun-dial to mark the hours; and 'marker' or 'pointer' is the literal meaning of the word. It connoted perpendicularity; hence we find Oenopides of Chios calling a perpendicular a straight line drawn 'gnomon-wise' (κατὰ γνώμονα). On a like ground the term is used in Theognis to describe an instrument for drawing right angles, like a carpenter's square. The transition was natural to the figure which remains over in a square when a smaller square is cut out of it (or the figure which, as Aristotle says, when added to a square preserves the shape and makes a larger square).



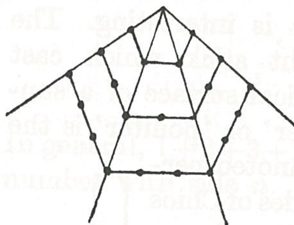
In Euclid, Book II, it has a slightly wider meaning, covering not only the gnomon which is part of a square, but the figure which is similarly related to any parallelogram. Later still, Heron of Alexandria defines a gnomon as that which, when added to anything, number or figure, makes the whole similar to that to which it is added.

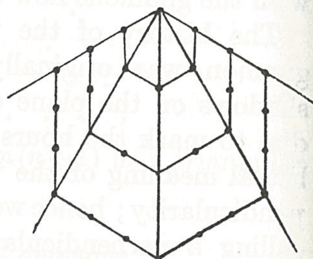


(c) Gnomons of the polygonal numbers.

In accordance with the generalized meaning of the word 'gnomon', all the polygonal numbers, like triangular and square numbers, are formed by adding gnomons successively to 1, which is potentially the first polygonal number of any form. The method of formation in the case of the pentagonal and hexagonal numbers is shown in the

accompanying figures, and it is seen that the successive gnomons are in the case of the pentagon 4, 7, 10 . . . with





the common difference 3, and in the case of the hexagon 5, 9, 13 . . . with the common difference 4. In general, the successive gnomonic numbers for any polygonal number, say, of n sides have $n-2$ for their common difference.

(*d*) *Right-angled triangles with sides in rational numbers.*

Pythagoras was aware that, while $3^2+4^2=5^2$, any triangle with its sides in the ratio of the numbers 3, 4, 5 is right angled. It would be natural to inquire whether there were other squares besides 5^2 which are the sum of two squares, or, in other words, to seek other sets of three integral numbers which can be made the sides of right-angled triangles; and here we have the beginning of the *indeterminate analysis* which culminated in the work of Diophantus. Since the sum of any number of successive terms of the series 1, 3, 5, 7 . . . is a square, it is only necessary to pick out of this series the odd numbers which are themselves squares; for if we take one of these, say 9, the addition of this square to the sum of the numbers which precede it in the series (which sum is a square) makes the square number which is the sum of the terms of the series up to and including 9. It would be natural to seek a formula which should enable all sets of three

numbers of this kind to be written down, and such a formula is actually attributed to Pythagoras. This formula amounts to the statement that, if m be any odd number,

$$m^2+\{\tfrac{1}{2}(m^2-1)\}^2=\{\tfrac{1}{2}(m^2+1)\}^2.$$

Pythagoras would presumably arrive at this rule of formation in this way. Observing that the gnomon put round the figure representing n^2 is $2n+1$, he would have to make $2n+1$ a square.

If we suppose that $2n+1=m^2$,

we obtain $n=\tfrac{1}{2}(m^2-1)$,

and therefore $n+1=\tfrac{1}{2}(m^2+1)$.

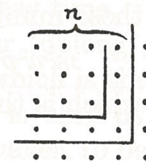
It follows that

$$m^2+\{\tfrac{1}{2}(m^2-1)\}^2=\{\tfrac{1}{2}(m^2+1)\}^2.$$

Another formula devised for the same purpose is attributed to Plato, namely

$$(2m)^2+(m^2-1)^2=(m^2+1)^2.$$

This is not equivalent to the result of multiplying Pythagoras' formula by 4 throughout, because m need not here be odd, as it must in Pythagoras' formula. But it may have been obtained, like the other, by consideration of gnomons in a figure. Consider the square with n dots in its side in relation both to the next smaller square, $(n-1)^2$, and to the next larger, $(n+1)^2$. Then n^2 exceeds $(n-1)^2$ by the gnomon $2n-1$, but falls short of $(n+1)^2$ by the gnomon $2n+1$. Therefore the square $(n+1)^2$ exceeds the square $(n-1)^2$ by the sum of the two gnomons $2n-1$ and $2n+1$, which is $4n$.



That is, $4n+(n-1)^2=(n+1)^2$,

and, substituting m^2 for n in order to make $4n$ a square, we have the Platonic formula

$$(2m)^2+(m^2-1)^2=(m^2+1)^2.$$

The formulae of Pythagoras and Plato supplement each other. Euclid's solution of the problem (Eucl. X, Lemma following Prop. 28) is more general, amounting to the following.

If AB be a straight line bisected at C and produced to D, then (Eucl. II. 6)

$$AD \, . \, DB+CB^2=CD^2,$$

which we may write thus,

$$uv=c^2-b^2,$$

where $u=c+b, \; v=c-b,$

and consequently

$$c=\tfrac{1}{2}(u+v), \; b=\tfrac{1}{2}(u-v).$$

Now, says Euclid, in order that uv may be a square, both u and v must either be squares, or 'similar plane numbers', and further they must be either both odd or both even in order that b and c may be whole numbers. 'Similar plane numbers' are of course numbers which are the product of two factors proportional in pairs, as $mp \, . \, np$ and $mq \, . \, nq$, or mnp^2 and mnq^2. Provided, therefore, that these numbers are both even or both odd,

$$m^2n^2p^2q^2+\{\tfrac{1}{2}(mnp^2-mnq^2)\}^2=\{\tfrac{1}{2}(mnp^2+mnq^2)\}^2,$$

which is the formula required.

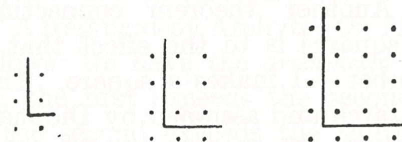
(e) *Oblong numbers.*

Just as the sum of any number of terms of the series of natural numbers 1, 2, 3 . . . is a triangular number, and the sum of any number of terms of the series of odd

numbers 1, 3, 5 . . . is a square, so the sum of any number of terms of the series of even numbers, 2, 4, 6 . . . is an 'oblong' number (ἑτερομήκης) with sides or factors differing by 1. In fact the sum of the series

$$2+4+6 \ldots +2n=n(n+1).$$

It cannot be doubted that the earliest Pythagoreans, if not Pythagoras himself, discovered this fact. They would also observe that the 'oblong' number is double of a triangular number. These facts would be brought out by taking two dots representing 2, and then placing round them, as successive gnomons, dots representing the even numbers 4, 6 . . . thus:



The successive oblong numbers are

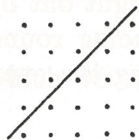
$$1 \, . \, 2=2, \quad 2 \, . \, 3=6, \; 3 \, . \, 4=12, \ldots n(n+1) \ldots$$

No two of these numbers are 'similar', for the ratio $n:(n+1)$ is different for all different values of n. As Aristotle says, the addition of each of these gnomons (4, 6, 8 . . .) changes the shape of the figure every time.

It should be noted that the Greek word for 'oblong', ἑτερομήκης, would literally cover any number which is the product of any two unequal numbers, and it is used by Plato and Aristotle in this sense; it came, however, to be restricted to numbers the factors of which differ by 1, i.e. numbers of the form $n(n+1)$. In Theon of Smyrna and Nicomachus a number of the form $m(m+n)$ where $n>1$ is called 'prolate' (προμήκης).

It is obvious that any oblong number $n(n+1)$ is the sum of two equal triangular numbers,

and that any square number is the sum of two successive triangular numbers. The annexed diagrams illustrate these facts; the second is proved by the identity

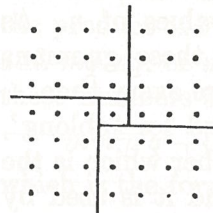


$$\tfrac{1}{2}n(n-1)+\tfrac{1}{2}n(n+1)=n^2.$$

Another theorem connecting triangular numbers and squares is to the effect that 8 times any triangular number + 1 makes a square. The theorem is quoted by Plutarch and assumed by Diophantus; it may easily go back to the Pythagoreans. In our notation

$$8\,.\,\tfrac{1}{2}n(n+1)+1=4n(n+1)+1=(2n+1)^2.$$

The truth of this theorem also would be obvious from a figure made up of dots in the usual way. The annexed figure shows a square with side 7 divided into four equal 'oblong' numbers with a single dot in the middle, and each of the four 'oblongs' gives two equal triangular numbers.



Speusippus and Philippus of Opus (fourth century B.C.) are said to have written on polygonal numbers; and Hypsicles, who wrote about 170 B.C., is twice mentioned by Diophantus as the author of a 'definition' of a polygonal number.

THE THEORY OF PROPORTION AND MEANS

In an often-quoted passage of his 'summary' Proclus says that Pythagoras discovered the theory of *proportionals* and the construction of the cosmic figures (the five regular solids); for it seems now to be generally agreed that we should read 'proportionals' (τῶν ἀνάλογον) instead of the variant 'irrationals' (τῶν ἀλόγων). The theory of means in particular was developed very early in the Pythagorean school in connexion with arithmetic and the theory of music. We are told that in Pythagoras' time there were three means, the arithmetic, the geometric, and the subcontrary, and that the name of the third, the subcontrary, was changed by Archytas and Hippasus to 'harmonic'. A fragment by Archytas *On Music* defines the three as follows: we have the *arithmetic* mean when, of three terms, the first exceeds the second by the same amount as the second exceeds the third; the *geometric* mean when the first is to the second as the second is to the third; the '*subcontrary* which we call *harmonic*' when the three terms are such that, by whatever part of itself the first exceeds the second, the second exceeds the third by the same part of the third. That is, if b is the harmonic mean between a and c, and if $a=b+\frac{a}{n}$, then $b=c+\frac{c}{n}$, whence in fact

$$\frac{a-b}{b-c}=\frac{a}{c},\ \text{or}\ \frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a}.$$

Philolaus is said to have called the cube a 'geometrical harmony' because it has 12 edges, 8 angles, and 6 faces, and 8 is, in harmonics, the mean between 12 and 6.

Iamblichus, after Nicomachus, mentions a special 'most perfect proportion', consisting of four terms and called

'musical', which was discovered by the Babylonians and first introduced into Greece by Pythagoras. It was, he says, used by Aristaeus of Croton, Timaeus of Locri, Philolaus and Archytas (among other Pythagoreans), and finally by Plato in his *Timaeus*. The proportion is

$$a : \frac{a+b}{2} = \frac{2ab}{a+b} : b,$$

a particular case being 12 : 9 = 8 : 6. The two middle terms are the arithmetic and harmonic means between the extremes.

The theory of means was further developed in the school, seven others being added from time to time to the first three, making ten in all. The fourth, fifth, and sixth are credited, at least in part, to Archytas and Hippasus, or alternatively to Eudoxus; the last four are said to have been added by two later Pythagoreans, Myonides and Euphranor.

The ten means are described by Nicomachus and Pappus, whose accounts only differ as regards one of the ten. They need not all be set out here. Their nature will be understood from the following explanation. If $a > b > c$, the following formulae show the first three means, the arithmetic, geometric, and harmonic.

	Formula.	Equivalent.
(1)	$\frac{a-b}{b-c} = \frac{a}{a} = \frac{b}{b} = \frac{c}{c}$	$a+c = 2b$ (arithmetic).
(2)	$\frac{a-b}{b-c} = \frac{a}{b}$	$ac = b^2$ (geometric).
(3)	$\frac{a-b}{b-c} = \frac{a}{c}$	$\frac{1}{a}+\frac{1}{c} = \frac{2}{b}$ (harmonic).

The next three are obtained by varying the right-hand

side in these formulae and equating $(a-b)/(b-c)$ to c/a, c/b, and b/a respectively. Three more are obtained by changing the left-hand side to $(a-c)/(a-b)$ and equating this to a/c, b/c, and a/b respectively; and a tenth arises from equating $(a-c)/(b-c)$ to b/c.

Geometric means.

Plato speaks in the *Timaeus* of the geometric means between two squares and two cubes, observing that between two 'planes' one such mean suffices, but to connect two 'solids' two means are necessary. This is equivalent to saying that, if p^2, q^2 are two square numbers,

$$p^2 : pq = pq : q^2,$$

while, if p^3, q^3 be two cube numbers,

$$p^3 : p^2q = p^2q : pq^2 = pq^2 : q^3,$$

the one mean in the first case being pq, and the two means in continued proportion in the second case being p^2q, pq^2. Nicomachus quotes the substance of Plato's remark as a 'Platonic theorem', and adds in explanation the equivalent of Eucl. VIII. 11, 12; but the theorem no doubt goes back to the Pythagoreans.

Boëtius has preserved in his *De inst. musica* a proof by Archytas of an interesting theorem about geometric means, namely that, if we have two numbers in the ratio known as ἐπιμόριος or *superparticularis*, i.e. in the ratio of $n+1$ to n, there can be no number which is a mean proportional between the two numbers. The theorem is Prop. 3 in the *Sectio Canonis* attributed to Euclid, and Archytas' proof is substantially identical with that in the *Sectio* (see pp. 136–7, *post*). This fact creates a presumption that there existed, at least as early as the date of Archytas (say 430–365 B.C.), an *Elements of Arithmetic* in the form which we call Euclidean.

THE IRRATIONAL

The subject of irrationals in general was for the Greeks a part of geometry rather than arithmetic, and necessarily so, because, for want of notation, an irrational of any sort could only be denoted by a straight line or a combination of lines. This is illustrated by Euclid's Book X on irrationals simple and compound; these are always irrational *straight lines*, and the whole treatment of the subject is geometrical. The first discovery of the existence of the irrational must, however, have been made as the result of arithmetical considerations or reasoning with numbers. It is certain that it was made with reference to the diagonal of a square in relation to its side, that is to say, the first irrational or incommensurable to be discovered was the equivalent of what we write as $\sqrt{2}$. The discovery can hardly have been made by Pythagoras himself, but it was certainly made in his school. The approximate date can only be conjectured. According to Plato's *Theaetetus*, Theodorus of Cyrene was the first to prove the irrationality of what we write as $\sqrt{3}$, $\sqrt{5} \ldots \sqrt{17}$; and we may infer that the irrationality of $\sqrt{2}$ had already been proved before Theodorus' time. Theodorus was Plato's teacher in mathematics, and the irrationality of $\sqrt{2}$ is unmistakably alluded to in the *Republic* as a thing well known, for there Plato speaks of 'the rational diameter of 5' (by '5' he means the square having 5 for its side) as being 7 ($= \sqrt{49}$), in contradistinction to the 'irrational diameter of 5' which is what we should write as $\sqrt{50}$, the 'rational diameter' (7) being an approximation to this. Again, there is a well-attested title of a work by Democritus (born 470 or 460 B.C.) *περὶ ἀλόγων γραμμῶν καὶ ναστῶν*, *On irrational straight lines and solids* (atoms); and it is difficult to resist

the conclusion that the irrationality of $\sqrt{2}$ was discovered before Democritus' time. The traditional proof of it, given by Aristotle and repeated in a proposition interpolated in Euclid, Book X, is by a *reductio ad absurdum* showing that, if the diagonal of a square is commensurable with its side, it will follow that one and the same number is both odd and even. The proof is substantially as follows.

Suppose the diagonal AC of a square to be commensurable with AB the side; and let $\alpha : \beta$ be their ratio expressed in the least possible numbers.

Then $\alpha > \beta$, and therefore α is necessarily > 1.

Now $$AC^2 : AB^2 = \alpha^2 : \beta^2,$$
and, since $AC^2 = 2AB^2$, $\alpha^2 = 2\beta^2$.

Hence α^2, and therefore α, is even.

Since $\alpha : \beta$ is a ratio in its lowest terms, it follows that β must be *odd*.

Let $\alpha = 2\gamma$; therefore $4\gamma^2 = 2\beta^2$ or $2\gamma^2 = \beta^2$, so that β^2, and therefore β, is *even*.

But β was also odd: which is impossible.

Therefore the diagonal AC cannot be commensurable with the side AB.

ALGEBRAIC EQUATIONS

(*a*) *The indeterminate equations of the second degree*
$$2x^2 - y^2 = \pm 1.$$

Not only did the Pythagoreans prove the incommensurability of $\sqrt{2}$ with 1, but they showed how to find any number of successive approximations to the value of $\sqrt{2}$ by finding any number of integral solutions of the above equations. The pairs of values of x, y were called 'side-' and 'diameter-' (diagonal-) 'numbers' respectively and, as the values increase, the ratio of y to x approximates more and more closely to $\sqrt{2}$. Theon of Smyrna explains

the formation of the series of 'side-' and 'diameter-' numbers thus. We will denote the successive 'diameters' by d_1, d_2 ... and the corresponding 'sides' by a_1, a_2 ... Theon begins by making $a_1=1$, $d_1=1$ (these values satisfy the above equation with the positive sign). The second 'side' and 'diameter' are formed from the first, the third from the second, and so on, according to the following scheme:

$$a_2=a_1+d_1, \quad d_2=2a_1+d_1,$$
$$a_3=a_2+d_2, \quad d_3=2a_2+d_2,$$
$$\cdots\cdots\cdots$$
$$a_{n+1}=a_n+d_n, \quad d_{n+1}=2a_n+d_n.$$

Since $a_1=d_1=1$, it follows that

$$a_2=1+1=\ 2, \quad d_2=2\,.\,1+1=\ 3,$$
$$a_3=2+3=\ 5, \quad d_3=2\,.\,2+3=\ 7,$$
$$a_4=5+7=12, \quad d_4=2\,.\,5+7=17,$$

and so on.

Theon states, with reference to these numbers, the general proposition that

$$d_n{}^2=2a_n{}^2\pm 1,$$

and observes (1) that the signs alternate as successive d's and a's are taken, $d_1{}^2-2a_1{}^2$ being equal to -1, $d_2{}^2-2a_2{}^2$ equal to $+1$, $d_3{}^2-2a_3{}^2$ equal to -1, and so on, while (2) the sum of the squares of *all* the d's will be double of the sum of the squares of all the a's. (If the number of successive terms in each series is finite, the number must be even.)

The properties stated depend on the truth of the identity

$$(2x+y)^2-2(x+y)^2=2x^2-y^2;$$

for, if x, y be numbers which satisfy one of the two

equations $2x^2-y^2=\pm 1$, the formula (if true) gives us two higher numbers, $(x+y)$ and $(2x+y)$, which satisfy the other equation.

Not only is the identity true, but we know how it was proved; for Proclus tells us, in his *Commentary on the Republic of Plato*, that Euclid proved it graphically in the second Book of the *Elements*, namely in II. 10, a proposition to the effect that, if AB is bisected in C and produced to D, then

$$AD^2+DB^2=2AC^2+2CD^2.$$

If $AC=CB=x$, and $BD=y$, this proposition gives

$$(2x+y)^2+y^2=2x^2+2\,(x+y)^2,$$

or

$$(2x+y)^2-2\,(x+y)^2=2x^2-y^2,$$

which is the formula required.

The general property is easily proved algebraically thus:

$$d_n{}^2-2\,a_n{}^2=(2\,a_{n-1}+d_{n-1})^2-2\,(a_{n-1}+d_{n-1})^2$$
$$=2\,a_{n-1}{}^2-d_{n-1}{}^2$$
$$=-(d_{n-1}{}^2-2\,a_{n-1}{}^2)$$
$$=+(d_{n-2}{}^2-2\,a_{n-2}{}^2), \text{ in like manner;}$$

and so on.

As the proof of the property is contained in Eucl. II. 10, it is a fair inference that that theorem is Pythagorean, and may even have been formulated for the specific purpose.

(β) The ἐπάνθημα ('bloom') of Thymaridas.

Thymaridas of Paros, an ancient Pythagorean, is famous as the author of a rule for solving a certain set of n simultaneous simple equations connecting n unknown quantities; so well known was it that it went by the name of the 'flower' or 'bloom' of Thymaridas. The rule is

stated in general terms, but the substance is pure algebra. The known quantities in the equations are ὡρισμένα (determinate), the unknown ἀόριστα (undetermined): cf. Diophantus' definition of his unknown as an 'undefined or undetermined number of units'.

The rule states in effect that, if we have the following n equations connecting the n unknowns $x, x_1 \dots x_{n-1}$,

$$x+x_1+x_2+\dots+x_{n-1}=s,$$
$$x+x_1=a_1,$$
$$x+x_2=a_2,$$
$$\cdot \quad \cdot \quad \cdot$$
$$x+x_{n-1}=a_{n-1},$$

the solution for x is

$$x=\frac{(a_1+a_2+\dots+a_{n-1})-s}{n-2}$$

Iamblichus, our informant, goes on to show that other types of equations can be reduced to this, so that the rule does not 'leave us in the lurch' in those cases either. He gives as an instance the indeterminate problem represented by the following three equations connecting four unknown quantities:

$$x+y=a(z+u),$$
$$x+z=b(u+y),$$
$$x+u=c(y+z).$$

From these equations we obtain

$$x+y+z+u=(a+1)(z+u)=(b+1)(u+y)=(c+1)(y+z).$$

If now x, y, z, u are all to be integers, $x+y+z+u$ must contain $a+1, b+1, c+1$ as factors. If L be the least common multiple of $a+1, b+1, c+1$, we can put

$x+y+z+u=L$, and we obtain from the above equations, in pairs,

$$x+y=\frac{a}{a+1}L,$$
$$x+z=\frac{b}{b+1}L,$$
$$x+u=\frac{c}{c+1}L,$$

while $x+y+z+u=L$.

These equations are of the type to which Thymaridas' rule applies, and, since the number of unknown quantities is 4, $n-2$ is in this case 2, and

$$x=\frac{L\left(\frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}\right)-L}{2}$$

The numerator is integral, but it may be an odd number, in which case we must substitute $2L$ for L as the value of $x+y+z+u$.

Iamblichus has the particular case where $a=2$, $b=3$, $c=4$. L is thus $3.4.5=60$, and the numerator of the above expression becomes $133-60$, or 73, an odd number; hence we take $2L$ or 120 in place of L, and so obtain $x=73$, $y=7$, $z=17$, $u=23$.

Iamblichus goes on to apply the method to the equations

$$x+y=\frac{3}{2}(z+u),$$
$$x+z=\frac{4}{3}(u+y),$$
$$x+u=\frac{5}{4}(y+z),$$

which give

$$(x+y+z+u)=\frac{5}{2}(z+u)=\frac{7}{3}(u+y)=\frac{9}{4}(y+z),$$

whence $x+y+z+u=\frac{5}{3}(x+y)=\frac{7}{4}(x+z)=\frac{9}{5}(x+u).$

In this case we take L, the least common multiple of 5, 7, 9, or 315, and put

$$x+y+z+u=L = 315,$$

so that $$x+y=\frac{3}{5}L=189,$$

$$x+z=\frac{4}{7}L=180,$$

$$x+u=\frac{5}{9}L=175;$$

therefore $$x=\frac{544-315}{2}=\frac{229}{2}.$$

In order that x may be integral, we take $2L$, or 630, instead of L, and the solution is $x=229$, $y=149$, $z=131$, $u=121$.

(γ) *Equation* $xy=2(x+y)$.

It would appear that the Pythagoreans considered the equivalent of this equation, for in the *Theologumena Arithmetices* it is observed that 16 is the only square the area of which is numerically equal to its perimeter. This corresponds to the solution $x=y=4$. If we write the equation in the form $(x-2)(y-2)=4$, we see that the integral solutions are obtained by equating $x-2$, $y-2$ to the respective factors of 4. These factors are (2, 2) and (4, 1). The only possible values of x, y are thus found to be (4, 4) or (3, 6). Therefore, besides the square, the only rectangle having the property in question is 3 . 6.

MANUALS OF 'ARITHMETIC'

The treatises on the Pythagorean theory of numbers which have survived are the work of Neo-Pythagoreans, though there is little in their content that does not go back to the immediate successors of Pythagoras. The first is the *Introductio arithmetica* of Nicomachus. Nicomachus of Gerasa, a city in Judaea beyond Jordan, flourished about A.D. 100. Of his life nothing is known. He wrote other works, one of which, the *Enchiridion Harmonices* or *Handbook of Harmony*, has survived. A *Theologumena Arithmetices*, or treatise on the theology or the mystic properties of numbers, is also attributed to him; this is not extant in its original form, but it cannot be doubted that the compilation with the same title which has come down to us, and which was edited by Ast along with the *Introductio* in 1817, contains extracts from the original work of Nicomachus as it does fragments from Speusippus and Anatolius, Bishop of Laodicea (A.D. 270). An *Introductio geometrica* is also mentioned by Nicomachus himself, who says in one place, with regard to certain solid numbers, that they were specially treated in it; but it does not necessarily follow that the treatise was his own.

The *Introductio arithmetica* with which we are here concerned deals in great part with the same subjects as the arithmetical Books (VII–IX) of Euclid's *Elements*. But the treatment is wholly different. In Euclid numbers are represented by straight lines with letters attached. Since a straight line can represent any number, the notation is as general as, though less concise than, our algebraical notation; and Euclid's proofs are general and scientific. In Nicomachus numbers are no longer denoted by straight

lines, so that, when undetermined numbers have to be expressed or distinguished, it is necessary to use ordinary language, which makes enunciations cumbrous and hard to follow. Nicomachus has little or nothing in the shape of proofs in the proper sense of the term. As a rule, he merely states a proposition, and then illustrates it by means of particular numbers. Mathematically speaking, therefore, Nicomachus' treatise (unlike Euclid's) has little scientific value. Indeed Nicomachus seems to have been a philosopher rather than a mathematician. His object was apparently to write a popular treatise which should arouse in the beginner an interest in the theory of numbers by making him acquainted with the more obvious properties which they had been proved to possess. But he himself was more interested in the mystic properties of numbers which appealed to the philosopher; hence his high-flown and rhetorical language when he is stating even the most obvious relations of numbers. It is difficult to account for the success of the work unless on the assumption that it was at first read by philosophers rather than mathematicians (Pappus evidently despised it), and afterwards became popular at a time when there were no mathematicians left, but only philosophers who incidentally took an interest in mathematics. But a success it was. This is shown by the fact that it was translated into Latin by Apuleius of Madaura (born about A.D. 125), and again by Boëtius, and that commentaries on it were written by Iamblichus, Heronas, Asclepius of Tralles, Johannes Philoponus, and Proclus. Its vogue is further attested by the allusion in Lucian's *Philopatris*, 'You calculate like Nicomachus', a remark which, in view of its context, seems like a jibe rather than a compliment. Both the *Introductio* and the commentary of Iamblichus have been edited in the Teubner series, the

former by Hoche and the latter by Pistelli. We have now an elaborate English translation by Martin Luther D'Ooge, edited, with studies in Greek arithmetic, by F. E. Robbins and L. C. Karpinski (University of Michigan Studies, vol. xvi, 1926).

Book I of the *Introductio* begins with generalities, partly philosophical, on the relation of the four subjects of the *quadrivium*, arithmetic, geometry, astronomy, and music. Then (c. 7) we have definitions of number, even and odd, next (cc. 8–10) the distinctions between three kinds of even numbers and (cc. 11–13) three kinds of odd numbers. Incidentally c. 13 explains the method of Eratosthenes' *sieve* (κόσκινον), a device for finding prime numbers. We set out the series of odd numbers beginning with 3,

$$3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25 \ldots$$

Now 3 is a prime number, but multiples of it are not; the multiples forming part of the series, namely 9, 15, 21 . . . are obtained by passing over two numbers at a time beginning from 3; we strike out these multiples as not being prime. Similarly 5 is a prime number, but, by passing over *four* numbers at a time beginning from 5, we obtain multiples of 5, namely 15, 25 . . .; these again we strike out for a like reason. In general, if n be a prime number, we find its multiples in the series by passing over $n-1$ terms at a time, and we strike out these multiples. When we have gone far enough with this process, the numbers which are still left will be primes. It is clear, however, that this primitive method would become useless as soon as we reached numbers of any considerable size. The same c. 13 gives the rule for finding out whether two numbers are prime to one another; it is the method of Eucl. VII. 1, equivalent to our rule for finding the greatest common

measure, but is described in general terms. The next chapters (cc. 14–16) discuss *perfect* numbers and distinguish them from *over-perfect* (ὑπερτελής) and *deficient* (ἐλλιπής) numbers. As regards Nicomachus' knowledge of perfect numbers see above (p. 41). An 'over-perfect' number is a number such that the sum of all its aliquot parts is greater, and a 'deficient' number a number such that the sum of all its aliquot parts is less, than the number itself.

Having considered numbers by themselves, Nicomachus next considers numbers in relation to one another. One number may be greater than, equal to, or less than, another. If it is greater, it may be greater, says Nicomachus, in one of five different ways according to the particular ratio (greater than 1) that it bears to it; and the reciprocals of these ratios give five corresponding different ways in which one number may be less than another.

1. The greater number may be a *multiple* (πολλαπλάσιος) of the lesser, in which case the lesser is a *submultiple* (ὑποπολλαπλάσιος) of the greater. Particular multiples are the *double* (διπλάσιος), *triple* (τριπλάσιος), &c.

2. If a, b are the greater and lesser numbers respectively, we may have

$$a = \left(1+\frac{1}{n}\right)b \text{ or } \frac{n+1}{n}b, \text{ so that } b = \frac{n}{n+1}a,$$

where n is any integer. These ratios are called respectively ἐπιμόριος or *superparticularis* (Boëtius) and ὑπεπιμόριος (*subsuperparticularis*). There were special names for the particular cases where n is 2, 3, &c.; when n is 2, the ratio is ἡμιόλιος (*sesquialter*), when n is 3 ἐπίτριτος (*sesquitertius*, &c.), and the reciprocals are the same with ὑπ(ό) or *sub* prefixed. ἐπίτριτος means of course 'one-third on', and so with the other cases.

3. If $a = \left(1+\frac{m}{n}\right)b$ $(m < n)$, and therefore $b = \frac{n}{n+m}a$, the ratios were called ἐπιμερής (*superpartiens*) and ὑπεπιμερής (*subsuperpartiens*) respectively.

Two subdivisions were distinguished in which

(1) $m = n-1$, so that $m/n = \frac{2}{3}, \frac{3}{4}$, &c.

(2) $m < n-1$, e.g. $m/n = \frac{2}{5}, \frac{3}{7}$, and the like.

4 and 5. In these cases the greater number contains, not *once* the number, but a *multiple* of it, plus a fraction. That is, $a = \left(p+\frac{1}{n}\right)b$, or $a = \left(p+\frac{m}{n}\right)b$, and there are subdivisions of the latter according as $m = n-1$ or $< n-1$. The word for the ratio $p+\frac{1}{n}$ is πολλαπλασιεπιμόριος, and that for $p+\frac{m}{n}$ is πολλαπλασιεπιμερής, and so on.

In c. 23 Nicomachus illustrates the fact that, if a, b, c be three numbers in geometrical progression with one of the above ratios as common ratio, three terms formed as follows, namely

$$a,\ a+b,\ a+2b+c,$$

or

$$c,\ c+b,\ c+2b+a,$$

are in geometrical progression with one or other of the above types of ratio as common ratio. Suppose, for example, that the original terms are a, na, n^2a. The first transformation gives three terms a, $a(n+1)$, $a(n+1)^2$ with common ratio $n+1$; the second gives three terms n^2a, $n(n+1)a$, $(n+1)^2a$, the common ratio being $(n+1)/n$; and so on.

Book II begins with a similar transformation. If a, b, c be three numbers in ascending geometrical progression with any common ratio other than 1, and if we form three numbers

$$a, \quad b-a, \quad c+a-2b,$$

we have three more terms in geometrical progression, but with a less common ratio. E.g. if the original terms are a, na, n^2a with n as common ratio, the transformation gives a, $(n-1)a$, $(n-1)^2a$, three terms in geometrical progression but with $n-1$ as common ratio; and by applying the process the necessary number of times we shall arrive at numbers in the ratio of equality.

C. 6 is preliminary to the subject of polygonal numbers and their respective gnomons. The numbers are shown graphically, with α's for dots, as above explained. There are no general proofs, but triangular numbers, squares, pentagonal, hexagonal, and heptagonal numbers are exhibited, and it is shown that the corresponding gnomons are the successive terms after 1 in the arithmetical progressions which have 1 for the first term and the numbers 1, 2, 3, 4, 5 respectively for common difference (cf. pp. 44–6 above). That is, the successive gnomons for triangles are 2, 3, 4 ..., for squares 3, 5, 7 ..., for pentagons 4, 7, 10 ..., and so on. By the aid of modern notation we may generalize thus. The gnomons for polygonal numbers of a sides are

$$1+a-2,\ 1+2(a-2),\ 1+3(a-2)\ldots$$

and an a-gonal number with n in its side is

$$1+\{1+(a-2)\}+\{1+2(a-2)\}+\ldots+\{1+(n-1)(a-2)\}$$
$$=n+\tfrac{1}{2}n(n-1)(a-2).$$

C. 12 mentions that a square is the sum of two consecutive triangular numbers, and that, if we add to any polygonal number a certain triangular number, we obtain a polygonal number with one more side. In fact, an a-gonal number of side n plus the triangular number of side $n-1$ gives the $(a+1)$-gonal number of side n, for

$$n+\tfrac{1}{2}n(n-1)(a-2)+\tfrac{1}{2}n1(n-)=n+\tfrac{1}{2}n(n-1)\{(a+1)-2\}.$$

In c. 13 Nicomachus passes to the first *solid* number, the *pyramid*. The base of the pyramid may be a triangular number, a square, or any polygonal number, but Nicomachus only mentions the first triangular pyramids 1, 4, 10, 20, 35, 56, 84, and explains (c. 14) the formation of pyramids on square bases. We may generalize as follows. An a-gonal number with n in its side is, as we have seen,

$$n+\tfrac{1}{2}n(n-1)(a-2).$$

It follows that the pyramidal number with that polygonal number for base is

$$1+2+3+\ldots+n+\tfrac{1}{2}(a-2)\{1.2+2.3+\ldots+(n-1)n\}$$
$$=\tfrac{1}{2}n(n+1)+\tfrac{1}{2}(a-2)\,.\,\tfrac{1}{3}(n-1)n(n+1).$$

In c. 14 Nicomachus speaks of pyramids *once*, *twice*, *or thrice truncated* (κόλουρος, δικόλουρος, τρικόλουρος). These are pyramids in which we have cut off (*a*) the unit at the top, (*b*) the unit and the next layer, (*c*) the unit and the next two layers respectively.

Other solid numbers are classified (cc. 15–17). Cubes are the products of three equal numbers; a *scalene* solid number is the product of three numbers all unequal. There were a variety of other names according to the relations between the factors: e.g. a *beam* (δοκίς) or *column* (στηλίς) has a square base while the height is greater than the side of the square, but in a *tile* (πλινθίς) the base is a square while the other edge is less than the side of the square. Cubes the last digit of which is the same as the last digit of the side (the sides and cubes in these cases end in 1, 5, or 6)

are called *spherical* or *recurrent* (ἀποκαταστατικοί); the squares of the same numbers end in the same digits and are called *circular* (κυκλικοί).

Sum of series of cube numbers.

In c. 20 we have the interesting statement that, if we set out the series of odd numbers

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19 \ldots,$$

the first, namely 1, is a cube; the sum of the next *two*, $3+5$, is a cube; the sum of the next *three*, $7+9+11$, is a cube; and so on. It is easy to deduce from these facts the summation of any number of terms of the series of natural cubes. In general, n^3 is the sum of n successive odd numbers beginning with a certain number determined by the fact that the number of terms preceding it in the above series is

$$1+2+3+\ldots+(n-1)=\tfrac{1}{2}(n-1)\,n.$$

Thus $1^3+2^3+3^3+\ldots+n^3$ is the sum of $\frac{1}{2}(n-1)\,n+n$, that is, $\frac{1}{2}n(n+1)$, terms of the series of odd numbers.

Therefore

$$1^3+2^3+3^3+\ldots+n^3=\tfrac{1}{4}n(n+1)\,[2+\{\tfrac{1}{2}n(n+1)-1\}\,2]$$
$$=\{\tfrac{1}{2}n(n+1)\}^2.$$

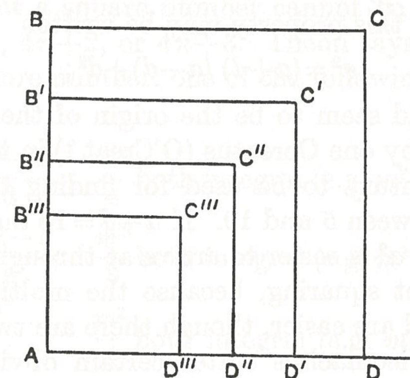
Nicomachus does not give this formula, but it was known to the Roman *agrimensores*, and it is not likely that Nicomachus was unaware of it. That it was a Greek discovery may fairly be inferred from the fact that al-Karkhī, the Arabian algebraist, who mainly followed Greek models, gives in his algebra entitled *al-Fakhrī* a proof by means of a figure with gnomons drawn about squares in the traditional Greek fashion.

Let AB be the side of a square AC; let

$$AB=1+2+\ldots+n=\tfrac{1}{2}n(n+1),$$

and along BA measure $BB'=n$, $B'B''=n-1$, $B''B'''=n-2$, and so on.

Draw the squares on AB', AB'' ... showing the gnomons which are the differences between the squares, as in the diagram.



Now (gnomon $BC'D$) $=BB'\,.\,BC+DD'\,.\,C'D'$

$$=BB'\,(BC+C'D').$$

But $BC=\frac{1}{2}n(n+1)$, and

$$C'D'=1+2+3+\ldots+(n-1)=\tfrac{1}{2}n(n-1).$$

Therefore (gnomon $BC'D$) $=n\,.\,n^2=n^3$.

Similarly (gnomon $B'C''D'$) $=(n-1)^3$, and so on.

Therefore $1^3+2^3+3^3+\ldots+n^3=$ the sum of the gnomons down to the gnomon about the small square at A which has 1 for its side, *plus* that small square itself.

That is, $1^3+2^3+3^3+\ldots+n^3=$ (square AC)

$$=\{\tfrac{1}{2}n(n+1)\}^2.$$

In cc. 21–9 Nicomachus deals with arithmetical and geometrical progressions and with means, namely the arithmetic, geometric, and harmonic, and the others already

mentioned (pp. 52–3). Various properties are mentioned in the first three cases; the rest are treated summarily.

If a, b, c be in arithmetical progression,

$$b^2 - ac = (a-b)(b-c) = (a-b)^2 = (b-c)^2.$$

We notice that, if the terms are written in the form $d+a$, a, $a-d$, this property may be written

$$a^2 = (a+d)(a-d) + d^2;$$

and this would seem to be the origin of the *regula Nicomachi* stated by one Ocreatus (O'Creat?) in the twelfth or thirteenth century to be used for finding the square of a number between 5 and 10. If $a+d=10$ and $a > d$, the theory is that a^2 is easier to arrive at through the formula than by direct squaring, because the multiplications on the right hand are easier, though there are two of them.

In c. 24 Nicomachus states certain obvious relations between three terms in descending geometrical progression and the common ratio. The same chapter cites the 'Platonic theorem' about the number of means in continued proportion between two square numbers and two cube numbers respectively, namely one in the first case and two in the second. Finally in c. 25 Nicomachus observes that, if $a > b > c$, then $a/b <=> b/c$ according as a, b, c are in arithmetical, geometrical, or harmonical progression.

The book by Theon of Smyrna purporting to be a manual of mathematical subjects such as a student would require to enable him to read Plato includes an arithmetical section treating the elementary theory of numbers on much the same lines as Nicomachus, though less systematically. It contains, however, two important things which are not found in Nicomachus. One is the account of the 'side-' and 'diameter-numbers' invented by the Pythagoreans for the purpose of finding the successive integral solutions of the equations $2x^2 - y^2 = \pm 1$ (see pp. 55–7 above). The other is a statement to the effect that, if m^2 is a square number, either m^2 or m^2-1 is divisible by 3, and again either m^2 or m^2-1 is divisible by 4. This is equivalent to saying that a square number cannot be of any of the forms $3n+2$, $4n+2$, or $4n+3$. Theon says further that, if m^2 is a square number, one of the following alternatives must hold:

(1) $\dfrac{m^2-1}{3}$, $\dfrac{m^2}{4}$ both integral (e.g. $m^2=4$),

(2) $\dfrac{m^2-1}{4}$, $\dfrac{m^2}{3}$ both integral (e.g. $m^2=9$),

(3) $\dfrac{m^2}{3}$, $\dfrac{m^2}{4}$ both integral (e.g. $m^2=36$),

(4) $\dfrac{m^2-1}{3}$, $\dfrac{m^2-1}{4}$ both integral (e.g. $m^2=25$).

We can hardly doubt that these discoveries were Pythagorean. The truth of the statements can easily be verified. Since any number must have one of the forms

$$6k, \quad 6k\pm 1, \quad 6k\pm 2, \quad 6k\pm 3,$$

any square must have one or other of the forms

$$36k^2, \quad 36k^2 \pm 12k+1, \quad 36k^2 \pm 24k+4, \quad 36k^2 \pm 36k+9,$$

and the consideration of these forms separately gives what is required to prove the facts stated.

Iamblichus, born at Chalcis in Coele-Syria, lived in the first half of the fourth century A.D. He was a pupil of Anatolius and Porphyry, and wrote nine books on the Pythagorean Sect with the following titles: I. On the Life of Pythagoras; II. Exhortation to Philosophy; III. On

mathematical science in general; IV. On Nicomachus' *Introductio arithmetica*; V–VII. On arithmetical science in physics, ethics, and theology respectively; VIII. On the Pythagorean geometry; IX. On the Pythagorean music. The first four of these books survive and are accessible in modern editions. We are here concerned with the Commentary on Nicomachus. This is an elaborate work, amounting almost to a revised edition with certain things added. Those which deserve notice are as follows.

A square is represented as a racecourse (δίαυλος) formed by the series of natural numbers beginning from 1 as the *start* (ὕσπληξ), going up to n as the turning-point (καμπτήρ), and then returning through $n-1$, $n-2$, &c., to 1, the *goal* (νύσσα). The sum is clearly made up of two triangular numbers with sides n, $n-1$ respectively, and is therefore equal to $\frac{1}{2}n(n-1)+\frac{1}{2}n(n+1)=n^2$.

Another proposition is of greater interest. Take any three consecutive numbers the greatest of which is divisible by 3. Add them, and we have a number consisting of a certain number of units, a certain number of tens, a certain number of hundreds, and so on. The numbers of each denomination are the *digits* of the number as we write it. Add the digits; this gives a smaller number. Add the digits of the smaller number, and so on. Then, says Iamblichus, the final result will always be the number 6. Take, e.g., the numbers 10, 11, 12; the sum is 33; add the digits and we have 6. Take again the numbers 994, 995, 996; the sum is 2985; the sum of the digits of this number is 24, and the sum of the digits of 24 is again 6.

Iamblichus, thirdly, gives the notice about the solution of the set of any number of simultaneous equations between as many unknowns which went by the name of the *Epanthema* of Thymaridas (see pp. 57–8 above).

IV

THE EARLIEST GREEK GEOMETRY. THALES

THE 'SUMMARY' OF PROCLUS

WE shall often, in the course of this history, have to refer to the so-called 'summary' of Proclus. This is contained in a few pages (65–70) of Proclus' *Commentary on Euclid, Book I*, and it reviews in the briefest possible outline the course of Greek geometry from the earliest times to Euclid, with special reference to the evolution of the Elements. It has often been called the 'Eudemian' summary, on the assumption that it is an extract from the great *History of Geometry* in four Books by Eudemus, the pupil of Aristotle. But a perusal of the actual summary suffices to show that it cannot have been written by Eudemus himself, though the earlier portion down to a certain sentence was probably based on material drawn from Eudemus' *History*. The sentence in question marks a break in the narrative.

'Those', it runs, 'who have compiled histories bring the development of this science to this point. Not much younger than these is Euclid, who put together the Elements, collecting many of the theorems of Eudoxus, perfecting many others by Theaetetus, and bringing to irrefragable demonstration the propositions which had only been somewhat loosely proved by his predecessors.'

Since Euclid was later than Eudemus, this could not have been written by Eudemus; on the other hand, the description of 'those who have compiled histories' suits Eudemus perfectly. There is, however, no such difference