Continuity

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0.1 Preliminaries: Distance and Norm

Let us recall the definition of **distance** in $\mathbb{R}^n$: the distance between two points (or vectors, depending on how we like to think of them) $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ is given by an analog of the Pythagorean theorem:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2} \tag{0.1}$$

(In $\mathbb{R}^2$ this is *exactly* the Pythagorean theorem.) Let us also recall the **norm** or **length** of a point/vector $\mathbf{x}$ in $\mathbb{R}^n$:

$$\|\mathbf{x}\| = d(\mathbf{x}, \mathbf{0}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{0.2}$$

Note that we can rephrase the distance in terms of the norm:

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| \tag{0.3}$$

Let us also recall the **triangle inequality**: If $\mathbf{x}$ and $\mathbf{y}$ are points/vectors in $\mathbb{R}^n$, then

$$\boxed{\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|} \tag{0.4}$$

and, what is a consequence of the triangle inequality, the **reverse triangle inequality**:

$$\boxed{\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \leq \|\mathbf{x} - \mathbf{y}\|} \tag{0.5}$$

The general proof of the triangle inequality usually employs the **Schwartz inequality**:

$$\boxed{|x_1 y_1 + \cdots + x_n y_n| \leq \|\mathbf{x}\|\|\mathbf{y}\|} \tag{0.6}$$

which can be rephrased in terms of the dot product as

$$\boxed{|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|} \tag{0.7}$$

(It would be a great exercise to prove the Schwartz inequality for yourselves. Try it!)

Let us prove the triangle inequality using the Schwartz inequality:

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\
&\leq \mathbf{x} \cdot \mathbf{x} + 2|\mathbf{x} \cdot \mathbf{y}| + \mathbf{y} \cdot \mathbf{y} \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2
\end{aligned}$$

and taking the square root gives the triangle inequality.

Let us now prove the reverse triangle inequality using the triangle inequality: By the triangle inequality we get the following two inequalities:

$$\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$$
$$\|\mathbf{y}\| = \|\mathbf{y} - \mathbf{x} + \mathbf{x}\| \leq \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x}\|$$

Since $\|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|$ *(Check this!) we have that*

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$$
$$\|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|$$

Subtracting $\|\mathbf{y}\|$ *from both sides of the first, and* $\|\mathbf{x}\|$ *from both sides of the second, we get that*

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$
$$-\left(\|\mathbf{x}\| - \|\mathbf{y}\|\right) \leq \|\mathbf{x} - \mathbf{y}\|$$

If we multiply the second inequality above by -1, *we get that*

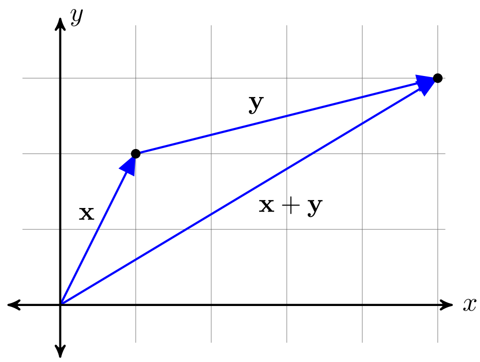
$$\|\mathbf{x}\| - \|\mathbf{y}\| \geq -\|\mathbf{x} - \mathbf{y}\|$$

Combining the first of the above two inequalities with this one gives

$$-\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

which is precisely the statement of the reverse triangle inequality. (The statement $|a| \leq b$ *is equivalent to* $-b \leq a \leq b$. *Check this!)*

Remark 0.1 *In* $\mathbb{R}^2$ *the triangle inequality is literally a triangle inequality, as you can see by drawing a picture of the triangle whose legs are* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{x} + \mathbf{y}$.



The length of $\mathbf{x} + \mathbf{y}$ *can only be shorter than the sum of the lengths of the two smaller legs,* $\mathbf{x}$ *and* $\mathbf{y}$. ■

0.2 Continuity of a Function at a Point

Now let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a (vector-valued) function. Recall that f sends points/vectors $\mathbf{x} = (x_1, \ldots, x_n)$ to points/vectors $\mathbf{y} = (y_1, \ldots, y_m)$, and since each component element y_i of $\mathbf{y}$ depends on how f treats the vector $\mathbf{x}$, y_i must be a real-valued function of $\mathbf{x}$,

$$\begin{aligned} &f_i : \mathbb{R}^n \to \mathbb{R} \\ &y_i = f_i(\mathbf{x}) \end{aligned} \tag{0.8}$$

We call f_i the i**th component function of** f. This means

$$\begin{aligned} \mathbf{y} &= (y_1, \ldots, y_n) \\ &= \big(f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})\big) \\ &= f(\mathbf{x}) \end{aligned} \tag{0.9}$$

of we can even more concisely write $f = (f_1, \ldots, f_m)$.

Example 0.2 *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by*

$$f(x_1, x_2, x_3) = (y_1, y_2) = (2x_1 - 3x_2 + x_3,\ -x_1 + x_2 + 2x_3)$$

Here, the component functions are given by

$$\begin{aligned} y_1 = f_1(x_1, x_2, x_3) = 2x_1 - 3x_2 + x_3 \\ y_2 = f_2(x_1, x_2, x_3) = -x_1 + x_2 + 2x_3 \end{aligned}$$

If we use different letters for the x_i and y_j, say we use $(u, v) = f(x, y, z)$, as is usually done in physics and engineering texts, then $u = f_1(x, y, z) = 2x - 3y + z$ and $v = f_2(x, y, z) = -x + y + 2z$. ■

Let us define appropriate notions of limit and continuity for such functions. We say that f has a **limit** in $\mathbb{R}^m$, which we denote $\mathbf{L} = (L_1, \ldots, L_m)$, at a point $\mathbf{a} = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$, and we write

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{L} \tag{0.10}$$

if for all $\varepsilon > 0$ we can find a $\delta > 0$ such that for all $\mathbf{x}$ different from $\mathbf{a}$ (but near $\mathbf{a}$), the following property is satisfied:

$$\text{If } \ d(\mathbf{x}, \mathbf{a}) < \delta, \text{ then } d\big(f(\mathbf{x}), \mathbf{L})\big) < \varepsilon. \tag{0.11}$$

We can rephrase property (0.11) using equation (0.3) as

$$\text{If } \ \|\mathbf{a} - \mathbf{x}\| < \delta, \text{ then } \|\mathbf{L} - f(\mathbf{x})\| < \varepsilon. \tag{0.12}$$

We say that $f : \mathbb{R}^n \to \mathbb{R}^m$ is **continuous at a point a** $= (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ if f has a limit $\mathbf{L}$ there and moreover that limit equals the value of f at $\mathbf{a}$, $\mathbf{L} = f(\mathbf{a})$. We can rephrase all of this in terms of ε's and δ's, if we wanted to, of course. If f is continuous at $\mathbf{a}$, then we write

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}) \tag{0.13}$$

Remark 0.3 *Let us analyze this definition. First of all, notice that we have complete freedom in choosing our positive number ε. The idea here is that in the range of f we can make the* *distance between $f(\mathbf{x})$ to $\mathbf{L}$ as small as we want, provided back in the domain of f we're within δ of our point $\mathbf{a}$, for* some *δ. That is, you give me a distance you want f to stay within from $\mathbf{L}$, and I can find you a neighborhood of $\mathbf{a}$ which is* entirely *mapped to within ε of $\mathbf{L}$ by f. The key point here is that the neighborhood of $\mathbf{a}$ may change depending on how small you want to keep the values of f from $\mathbf{L}$. If you make ε smaller, you'll probably have to shrink δ and so the neighborhood of $\mathbf{a}$.*

The picture you should have in mind is of a large lever. The lever is the function, the handle is the domain of the function, and the end on the other side of the fulcrum is the range of the function, that is the distance you can raise an object. If you push the lever a certain amount, you raise an object another given amount. The idea of continuity in such a situation is intuitive, namely the lever can't disappear at one place and reappear at another, and the way we formalize this intuition is by saying that if you want to look really really close to a point near the object you're trying to lift (within an ε), you'll see the end of the lever there so long as you only wiggle the handle no more than a given amount (your δ-neighborhood). There can be no skipping or disappearing and reappearing in such a situation, because if something like that happened, we could shrink our ε-window to a distance smaller than the size of the skip, and then the end of the lever would disappear if we wiggle the handle at all. ■

Remark 0.4 *Let us look at the case where $m = 1$, i.e. the case of a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$. In the case of $\mathbb{R}$, the norm reduces to the absolute value, for if $\mathbf{x} = (x_1)$, then $\|\mathbf{x}\| = \|(x_1)\| = \sqrt{x_1^2} = |x_1|$. Thus, the limit and continuity definitions can be rephrased using $|f(\mathbf{x}) - L|$ and $|f(\mathbf{x}) - f(\mathbf{a})|$, respectively. This is the definition found in your book.* ■

Example 0.5 *Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (2x - 3y + z,\ -x + y + 2z)$. Let us show that f is continuous at every point $\mathbf{a} = (a, b, c)$ in $\mathbb{R}^3$. Toward this end, let us note that for any $\mathbf{v} = (u, v, w)$ we have*

$$\begin{aligned}
\|f(\mathbf{v})\| &= \|f(u, v, w)\| \\
&= \|(2u - 3v + w,\ -u + v + 2w)\| \\
&= \sqrt{(2u - 3v + w)^2 + (-u + v + 2w)^2} \\
&\leq \sqrt{(2^2 + (-3)^2 + 1^2)(u^2 + v^2 + w^2) + ((-1)^2 + 1^2 + 2^2)(u^2 + v^2 + w^2)} \\
&\leq \sqrt{\left[2^2 + (-3)^2 + 1^2 + (-1)^2 + 1^2 + 2^2\right](u^2 + v^2 + w^2)} \\
&= \sqrt{2^2 + (-3)^2 + 1^2 + (-1)^2 + 1^2 + 2^2}\sqrt{u^2 + v^2 + w^2)} \\
&= \sqrt{20}\|\mathbf{v}\|
\end{aligned}$$

where the first inequality is the Schwartz inequality applied to each term being squared (i.e. $2u - 3v + w = (2, -3, 1) \cdot (u, v, w)$), and the second inequality is from adding more stuff and factoring. Consequently, since $f(\mathbf{x}) - f(\mathbf{a}) = f(\mathbf{x} - \mathbf{a})$, i.e. $f(x, y, z) - f(a, b, c) = f(x - a, y - b, z - c)$ (check this!), we have

$$\|f(\mathbf{x}) - f(\mathbf{a})\| = \|f(\mathbf{x} - \mathbf{a})\| \leq \sqrt{20}\|\mathbf{x} - \mathbf{a}\|$$

Thus, for any $\varepsilon > 0$ that we pick we can find our $\delta > 0$, namely $\delta = \frac{\varepsilon}{\sqrt{20}}$, for then, whenever $\|\mathbf{x} - \mathbf{a}\| < \delta$, we have

$$\|f(\mathbf{x}) - f(\mathbf{a})\| \leq \sqrt{20}\|\mathbf{x} - \mathbf{a}\| < \sqrt{20}\delta = \sqrt{20}\frac{\varepsilon}{\sqrt{20}} = \varepsilon$$

Voila! We have shown that $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$ for any $\mathbf{a} = (a, b, c)$ in $\mathbb{R}^3$. ■

Remark 0.6 *This example is a lot harder than what you are asked to do on homeworks or WebWork, but the method used in it is universally applicable. The basic idea is to try get an inequality of the form*

$$\|f(\mathbf{x}) - f(\mathbf{a})\| \leq C\|\mathbf{x} - \mathbf{a}\|$$

for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ for some positive constant C. If you get to this point, you're home free (but beware, it's not always possible; a good example is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1/x$). The intuition is clear: let $\mathbf{x}$ approach $\mathbf{a}$, then you squeeze $\|\mathbf{x} - \mathbf{a}\|$, which consequently squeezes $\|f(\mathbf{x}) - f(\mathbf{a})\|$ by a proportionate amount. Formally, say you want to squeeze $\|f(\mathbf{x}) - f(\mathbf{a})\|$ to under ε, then all you have to do is squeeze $\|\mathbf{x} - \mathbf{a}\|$ to under ε/C.

As a matter of fact, if you get an inequality of the type $\|f(\mathbf{x}) - f(\mathbf{a})\| \leq C\|\mathbf{x} - \mathbf{a}\|$, you basically succeeded in showing continuity. You can just say $\|f(\mathbf{x}) - f(\mathbf{a})\| \leq C\|\mathbf{x} - \mathbf{a}\| \to 0$ and you're done. Do this on exams! ■

Theorem 0.7 *If $f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$, then f is continuous if and only if each of the component functions $f_i : \mathbb{R}^n \to \mathbb{R}$ is continuous.* ■

The proof of this theorem is a simple consequence of the fact that

$$|x_i - a_i| \leq \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} = \|\mathbf{x} - \mathbf{a}\| \leq \sqrt{n} \max_{1 \leq i \leq n} |x_i - a_i|$$

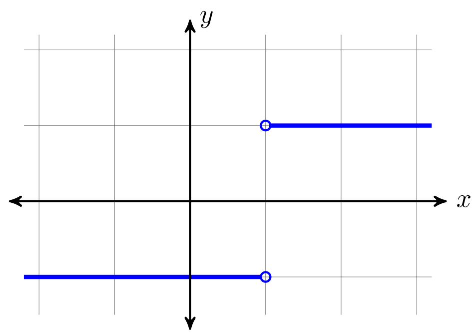
for all $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$ (Check this!) and then chasing the definition of continuity around. We omit the full tedious proof. The upshot of this very important theorem, however, is that we don't actually need to study the continuity of vector-valued functions to know what's going on. We need only study real-valued functions, for every vector-valued function is made up of real-valued component functions!

Theorem 0.8 *Sums, products, scalar multiples, and quotients (so long as the denominator does not go to 0) of continuous functions are continuous.* ■

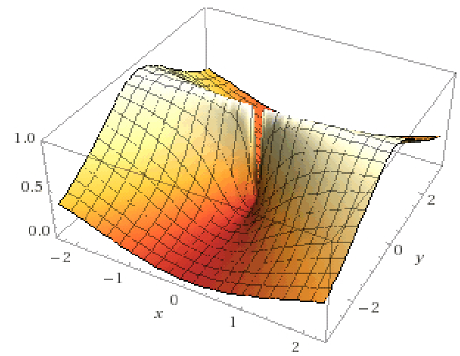
We also omit the proof of this theorem, as it's a lot of chasing ε's and δ's. I recommend taking the undergraduate analysis course for the full details!

Remark 0.9 *How does one show* ***discontinuity****? There are three basic reasons a function f may be discontinuous.*

1. *First, the limit may not exist. A single variable example of this type of problem is encountered with the function $f(x) = \frac{x-1}{|x-1|}$.*



The limit from the left and the limit from the right do not agree at $x = 1$. In two or more dimensions this problem manifests itself in even more varied ways, for the simple reason that a given point, say in $\mathbb{R}^2$, call it $\mathbf{a} = (a, b)$, may be approcated from more than two ways (not just from the left and the right). It can be approached along any path, even curved paths. A simple example of this is $f(x, y) = \frac{x^2}{x^2+y^2}$. Approaching $(0, 0)$ along the line $x = 0$, we see that f has a constant value of 0, while approaching $(0, 0)$ along the line $y = x$ we see that f has a constant value of $1/2$. These two are not the same z-values, so f cannot even have a limit at $(0, 0)$, even if we defined it to have some value there. You can see the rupture in the graph of f:



2. *Second, the limit may exist, but the value of the function may not be equal to that of the limit. For example,*

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0) \\ 1, & \text{if } (x, y) = (0, 0) \end{cases}$$

The limit as (x, y) approaches $(0, 0)$ is 0, but the value of f at $(0, 0)$ is 1. This is analogous to the many Calc 1 examples, such as

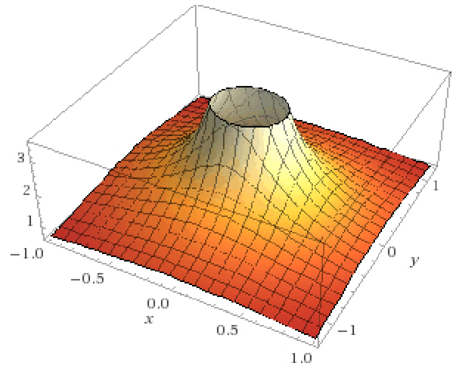
$$f(x) = \begin{cases} x^2, & \text{if } x \neq 2 \\ 1, & \text{if } x = 2 \end{cases}$$

This is called a **removable discontinuity**. *Let us prove that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. This is easy, for we have the inequality*

$$|f(x, y) - 0| = \left| \frac{x^2 y}{x^2 + y^2} \right| = \frac{x^2|y|}{x^2 + y^2} \leq |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} = \|(x, y) - (0, 0)\|$$

on account of $\frac{x^2}{x^2+y^2} \leq 1$ for all (x, y) in $\mathbb{R}^2$. By the above Remark 0.6 we are done, for $\|(x, y) - (0, 0)\| \to 0$ by assumption.

3. *Third, the function is not defined there and/or escapes to infity or negative infinity. For example, $f(x, y) = 1/\sqrt{x^2 + y^2}$ as (x, y) goes to $(0, 0)$. This is exactly the two dimensional analog of the function $f(x) = 1/|x|$. You can see this by noting that $\sqrt{x^2 + y^2} = \|(x, y)\|$, so the function is $f(\mathbf{x}) = 1/\|\mathbf{x}\|$, in fact.* ■



0.3 Further Examples

Example 0.10 *$f(x,y) = \dfrac{e^{\sin x}}{\cos y}$ is continuous on the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, \frac{\pi}{4}]$, because $\cos y$ is nonzero and continuous on $[0, \frac{\pi}{4}]$, and it's reciprocal is nonzero there, and also $\sin x$ and therefore $e^{\sin x}$ is continous, so their quotient is continuous.* ■

Example 0.11 *$f(x,y) = \dfrac{x+y}{x-y}$ is not continuous at $(0,0)$, for if we approach $(0,0)$ along the line $x = -y$, we get $f(x,y) = f(-y,y) = \frac{-2y}{-2y} = -1$, while if we approach $(0,0)$ along the line $x = 2y$ we get $f(x,y) = f(2y,y) = \frac{3y}{y} = 3$.* ■

Example 0.12 *$f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ is not continuous at $(0,0)$, for if we approach $(0,0)$ along the line $x = y$ we get that $f(x,y) = f(x,x) = 0$, while if we approach $(0,0)$ along the line $x = 2y$ we get that $f(x,y) = f(2y,y) = \frac{3y^2}{5y^2} = \frac{3}{5}$.* ■

Example 0.13 *$f(x,y) = \dfrac{xy}{|xy|}$ is not continuous at $(0,0)$. Taking $x, y > 0$, then $f(x,y) = 1$, while taking $x > 0$ and $y < 0$, say, we have $f(x,y) = -1$.* ■

Example 0.14 *$f(x,y) = \dfrac{x^2}{x^2+y}$ is not continuous at $(0,0)$, for look at the curves $y = 0$, $y = x^2$ and $y = 2x^2$. f has values 1, 1/2 and 1/3, respectively, on those curves as it approaches $(0,0)$.* ■

Example 0.15 *The function $f(x,y) = 3x - 2y + 5$ is continuous everywhere, for by the triangle inequality*

$$\begin{aligned}
|f(x,y) - f(a,b)| &= |(3x - 2y + 5) - (3a - 2b + 5)| \\
&= |3(x-a) - 2(y-b)| \\
&\leq |3(x-a)| + |-2(y-b)| \\
&= 3|x-a| + 2|y-b| \\
&= 3\sqrt{(x-a)^2} + 2\sqrt{(y-b)^2} \\
&\leq 3\sqrt{(x-a)^2 + (y-b)^2} + 2\sqrt{(x-a)^2 + (y-b)^2} \\
&= (3+2)\sqrt{(x-a)^2 + (y-b)^2} \\
&= 5\|(x,y) - (a,b)\| \to 0
\end{aligned}$$