Calculus I-II Review Sheet

1 Definitions

1.1 Functions

A function is f is **increasing** on an interval if $x \leq y$ implies $f(x) \leq f(y)$, and **decreasing** if $x \leq y$ implies $f(x) \geq f(y)$. It is called monotonic if it is either increasing or decreasing on its entire domain.

Example 1.1 *e^x is increasing on $(-\infty, \infty)$ and $\ln x$ is increasing on $(0, \infty)$. $f(x) = 1/x$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. The sine function is increasing on $[-\pi/2, \pi/2]$, $[3\pi/2, 5\pi/2]$, etc.* ■

A function $f(x)$ is **proportional** to another function $g(x)$ if there is a nonzero constant k such that $f(x) = kg(x)$.

Example 1.2 *The force of gravity F between two masses M and m is proportional to the inverse square function of the distance between them, r, namely*

$$F = \frac{GMm}{r^2}$$

Here $k = GMm$ and $g(r) = 1/r^2$. ■

The graph of a function f is **concave up** if it bends upward as we move from left to right, that is if, when we pick two points (x_1, y_1) and (x_2, y_2) on the graph of f, the graph lies below or on the line segment joining those points on that inteval. Similarly, the graph of f is **concave down** if it bends downward, or if the graph of f lies above or on any line segment joining two points on its graph. An **inflection point** is a point $(x, f(x))$ on the graph of f at which the graph changes concavity (or equivalently just the x value of that point).

Example 1.3 *The cosine function is concave up on $[\frac{\pi}{2}, \frac{3\pi}{2}]$ and concave down on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, etc. It has inflection points at odd multiples of $\pi/2$.* ■

A function f is said to be **even** if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$. Even functions are symmetric about the y-axis, while odd ones are symmetric about the origin.

Example 1.4 *The cosine function is even, $f(x) = |x|$ is even, and $f(x) = e^{3x^2}$ is even, while the sine function is odd, $f(x) = x^3$ is odd, and $f(x) = \arctan(x)$ is odd.* ■

1.2 Limits and Continuity

Suppose f is defined on an interval around a given real number c, except perhaps at $x = c$ itself. The **limit** of f as x approaches a given real number c is a real number L satisfying the following condition: for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$|x - c| < \delta \qquad \text{and} \qquad x \neq c$$

implies

$$|f(x) - L| < \epsilon$$

In this case we write

$$\lim_{x \to c} f(x) = L$$

Remark 1.5 *In plain English, this means that the limit L exists provided the following holds: however small I want to make my ϵ-window on the y-axis around L, I can find a small enough δ-window on the x-axis around c which is mapped by f entirely into the ϵ-window.* ■

Remark 1.6 *Note the logical implications: I have freedom to choose ϵ. I can choose it to be one in a billion, whatever I want. But once chosen, I'm typically restricted in my choice of δ. For example, if $f(x) = x^2$ and $L = 100$, then clearly $c = 10$, and to show that the limit as x approaches 10 is 100, I need to find a δ for any given ϵ. Suppose I've chosen my ϵ, say $\epsilon = 1/1000$. Then you can check that a good enough δ is $\delta = 1/21000$. It's not the only one, since I can pick it to be smaller than this, but I am restricted by how big I can make it. It can't be much bigger than that.* ■

Remark 1.7 *Note also that the definition of limit includes absolute values around $x - c$. We write $|x - c|$ and this implies the existence and equality of the **left** and **right limits**,*

$$\lim_{x \to c^-} f(x) \qquad \textit{and} \qquad \lim_{x \to c^+} f(x)$$

respectively. ■

Example 1.8 *The function $f(x) = \frac{x-1}{|x-1|}$ has left and right limits as x approaches 1, but the two limits are not equal. Hence f has no limit at $x = 1$.* ■

We can analogously define

$$\lim_{x \to \pm\infty} f(x)$$

The difference is we can't use the δ part of the definition. The ϵ part is the same, we can choose any $\epsilon > 0$ such that $|f(x) - L| < \epsilon$ under some appropriate condition on x. It's just that condition can't be the existence of δ such that $|x - c| < \delta$ (because $c = \infty$ and $|x - \infty| = \infty$, which is never less than δ), so instead we demand the existence of an $M > 0$ such that $x \geq M$ implies $|f(x) - L| < \delta$.

A function f is **continuous at a point** $x = c$ if the limit $\lim_{x \to c} f(x) = L$ exists and moreover $L = f(c)$, i.e. if

$$\lim_{x \to c} f(x) = f(c)$$

We say f is **continuous on an interval** $[a, b]$ or (a, b), etc., if it is continuous at all points in that interval.

Example 1.9 *All polynomials are continuous on $(-\infty, \infty)$. All rational functions $p(x)/q(x)$ are continuous except at zeros of $q(x)$. For example, $\frac{2x^2+3}{x^2-5}$ is continuous everywhere except at $\pm\sqrt{5}$. The exponential function e^x is continuous on $(-\infty, \infty)$, the log function $\ln x$ is continuous on $(0, \infty)$. The sine and consine functions are continuous on $(-\infty, \infty)$. The absolute value function $f(x) = |x - 2|$ is continuous everywhere on $(-\infty, \infty)$.* ■

Discontinuities, or **points of discontinuity** of f, are x-values where f is not continuous. For example 0 is a point of discontinuity of $\csc x$ and of $f(x) = 1/x$. A **removable discontinuity** is one that can be "plugged", e.g. $x = -2$ is a removable discontinuity of $f(x) = \frac{x^2-4}{x+2}$. An **essential discontinuity** is one that cannot be "plugged", typically because f goes to $\pm\infty$ near that point, for example $x = 1$ is an essential discontinuity of $f(x) = 2/(x-1)$. A **jump discontinuity** is a discontinuity where f "jumps" a finite amount near it, for example $x = 1$ for $f(x) = \frac{x-1}{|x-1|}$.

1.3 Rate of Change and the Derivative

The **average rate of change** of a function f between $x = a$ and $x = b$ is defined to be the slope of the line connecting the points $(a, f(a))$ and $(b, f(b))$ on the graph of f:

$$\text{average rate of change } = \frac{f(b) - f(a)}{b - a}$$

If we define $h = b - a$, then $b = a + h$, and the average rate of change between a and $b = a + h$ becomes the **difference quotient**:

$$\frac{f(a+h) - f(a)}{h}$$

If the limit of the difference quotient as h approaches 0 exists, then we say that f is **differentiable at** $x = a$ and we call this limit the **derivative of** f **at** a. It is a real number, denoted equivalently by

$$f'(a) \equiv \frac{df}{dx}(a) \equiv \left.\frac{dy}{dx}\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

If f is differentiable on an entire interval, then varying the point a, in other words letting x vary, we get a function, the **derivative function** $f'(x)$. One of our tasks is to find various derivative functions for well-known functions.

Example 1.10 *Common derivative functions are the following:*

1. $(x^n)' = nx^{n-1}$ *(power rule)*
2. $(\sin x)' = \cos x$
3. $(\cos x)' = -\sin x$
4. $(\tan x)' = \sec^2 x$
5. $(e^x)' = e^x$
6. $(\ln x)' = \frac{1}{x}$
7. $(a^x) = (\ln a)a^x$
8. $(\arctan x)' = \frac{1}{1+x^2}$
9. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ ∎

Remark 1.11 *To compute more complicated derivatives, such as those of xe^{x^2}, for example, we will need to know how to break up the process of taking a derivative into several steps involving only derivatives of things we know how to compute, such as (1)-(6) above. For example, we know how to take the derivative of x, x^2 and e^x, and we will develop methods to compute $(xe^{x^2})'$ in terms of these easier ones. This is the content of the sum, product, quotient and chain rules below.* ■

The derivative $f'(a)$ at $x = a$ can be interpreted as the **slope of the tangent line** to the graph of f at the point $(a, f(a))$. Since we have a slope, $f'(a)$, and a point, $(a, f(a))$, we can find the equation of the tangent line using point-slope:

$$y - f(a) = f'(a)(x - a)$$

Adding $f(a)$ to both sides gives the **equation for the tangent line**, which we also call the **linear approximation to f near** $x = a$ or **local linearization near** $x = a$:

$$y = f(a) + f'(a)(x - a)$$

This is the key idea behind the idea of a derivative, to approximate a complicated function $f(x)$ with the simplest one possible, a line, at least locally. *The tangent line is an honest-to-goodness approximation* of f near $x = a$, because we know that as $x \to a$, $\frac{f(x)-f(a)}{x-a} \to f'(a)$, i.e. from the definition of limit we get

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} \approx f'(a) \quad &\iff \quad f(x) - f(a) \approx f'(a)(x - a) \qquad &&\text{multiply both sides by } (x - a) \\ &\iff \quad f(x) \approx f(a) + f'(a)(x - a) \qquad &&\text{add } f(a) \text{ to both sides} \end{aligned}$$

Example 1.12 *Let us approximate the value of* $f(x) = e^{\sin x}$ *near* $x = \pi$ *using local linearization, say at* $x = 3$ *which is near* $x = \pi$. *Since* $f'(x) = e^{\sin x} \cdot \cos x$, *we have* $f'(\pi) = e^{\sin \pi} \cdot \cos \pi = -1$, *and because* $f(\pi) = e^{\sin \pi} = 1$, *we get*

$$\begin{aligned} f(x) &\approx f(\pi) + f'(\pi)(x - \pi) \\ &= 1 - (x - \pi) \\ &= 1 + \pi - x \end{aligned}$$

Therefore,

$$f(3) \approx 1 + \pi - 3 = \pi - 2 \approx 1.14$$ ■

The **error** in the tangent/linear approximation is the difference between the actual value $f(x)$ at x and the approximate value obtained from the tangent line at x near $x = a$:

$$E(x) = f(x) - \big[f(a) + f'(a)(x - a)\big]$$

It is a theorem, not hard to prove, that

$$\lim_{x \to a} \frac{E(x)}{x - a} = 0$$

It is also a fact, not proven in the book until the section on Taylor's theorem, that

$$E(x) \approx \frac{f''(a)}{2}(x - a)^2$$

Example 1.13 *The error in the previous example is*

$$E(x) = f(x) - (1 + \pi - x) = e^{\sin x} + x - 1 - \pi$$

so, for example our approximation at $x = 3$ *is off by about*

$$E(3) = e^{\sin 3} + 3 - 1 - \pi \approx 0.00997$$

Not bad! And, to complete this example, note that

$$\frac{f''(\pi)}{2}(3-\pi)^2 = \frac{1}{2}(\cos^2 3e^{\sin 3} - (\sin 3)e^{\sin 3})(3-\pi)^2 \approx 0.00968$$

which is indeed close to $E(3)$. ■

1.4 Optimization

A **critical point** of f is a point $x = a$ where $f'(a) = 0$, that is critical points are zeros of the derivative of f. Hence, when looking for them you have to solve the equation $f'(x) = 0$ for x. The y-value of f at a critical point $y = f(a)$ is called a **critical value**. Note that $f(a) \neq 0$ in general—it's $f'(a)$ that equals 0!

A **local minimum** of f is a point $x = a$ such that $f(x) \geq f(a)$ near a, and a **local maximum** of f is a point $x = a$ such that $f(x) \leq f(a)$ near a. These are upgraded to global minimum and global maximum if $f(x) \geq f(a)$ or $f(x) \leq f(a)$, respectively, for *all* x in the domain of f under consideration, not just those nearby.

1.5 The Riemann Integral

Let $[a, b]$ be a nonempty closed interval in $(-\infty, \infty)$ and let $f : [a, b] \to (-\infty, \infty)$ be bounded and continuous. Let $P = (a = x_0 < x_1 < \cdots < x_n = b)$ be a partition of $[a, b]$, that is an increasing finite increasing sequence of points in the interval. For a given partition P choose, for each $i = 0, 1, \ldots, n$, an arbitrary t_i in $[x_{i-1}, x_i]$. The **Riemann Sum** of f for a given partition P is then defined as

$$S_P = \sum_{i=1}^{n} f(t_i) \cdot (x_i - x_{i-1})$$

If we choose the x_i such that $x_n - x_{n-1} = x_{n-1} - x_{n-2} = \cdots = x_1 - x_0$, that is if

$$\Delta x = \frac{b-a}{n} \qquad \text{and} \qquad x_i = a + i\Delta x$$

then the Riemann sum is written simply as

$$S_P = \sum_{i=1}^{n} f(t_i)\Delta x$$

If we choose the left endpoint every time, $t_i = x_i$, then we get the **left Riemann sum**,

$$S_L = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

and if we choose the right endpoint, $t_i = x_i + 1$, then we get the **right Riemann sum**,

$$S_R = \sum_{i=1}^{n} f(x_i)\Delta x$$

Similarly, we have the **upper** and **lower Riemann sums**, U and L, obtained by choosing t_i in $[x_i, x_{i+1}]$ such that $f(t_i) = \max_{x_i \le t \le x_{i+1}} f(t)$ and $f(t_i) = \min_{x_i \le t \le x_{i+1}} f(t)$, respectively (since f is continous, we can find such t_i by the Extreme Value Theorem!). We can also choose t_i to be the midpoint of $[x_i, x_{i+1}]$, in which case we obtain the **midpoint Riemann sum**.

Remark 1.14 *Note that U does not in general equal S_L or S_R, and likewise with L. A good example is the upper half of the circle, $f(x) = \sqrt{1-x^2}$ on $[-1,1]$. Try to compute S_L, S_R, U and L.* ■

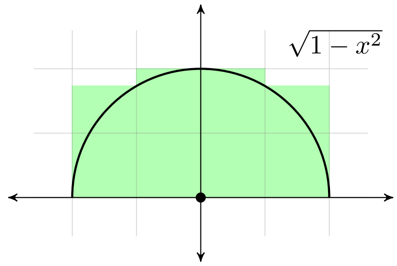
Example 1.15 *Let us compute the Riemann sum for $f(x) = \sqrt{1-x^2}$ on $[-1,1]$, and so obtain several approximations of the upper half of the unit circle. Let us choose $n = 4$ for convenience, which means $x_0 = -1$, $x_1 = -\frac{1}{2}$, $x_2 = 0$, $x_3 = \frac{1}{2}$, and $x_4 = 1$, and so $P = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$ is our partition of $[-1,1]$. Let us compute the left Riemann sum first: this means choosing $t_i = x_i$, and since $\Delta x = \frac{1-(-1)}{4} = \frac{1}{2}$, we have*

$$\begin{aligned}
S_L &= \sum_{x=0}^{4-1} f(x_i)\Delta x \\
&= f(-1)\cdot\frac{1}{2} + f(-\frac{1}{2})\cdot\frac{1}{2} + f(0)\cdot\frac{1}{2} + f(\frac{1}{2})\cdot\frac{1}{2} \\
&= \left(\sqrt{1-(-1)^2} + \sqrt{1-(-1/2)^2} + \sqrt{1-0^2} + \sqrt{1-(1/2)^2}\right)\cdot\frac{1}{2} \\
&= \left(0 + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2}\right)\cdot\frac{1}{2} \\
&= \frac{\sqrt{3}+1}{2}
\end{aligned}$$

Similarly, by choosing $t_i = x_{i+1}$, we get the right sum,

$$\begin{aligned}
S_R &= \sum_{x=0}^{4-1} f(x_{i+1})\Delta x \\
&= f(-1/2)\cdot\frac{1}{2} + f(0)\cdot\frac{1}{2} + f(1/2)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} \\
&= \left(\sqrt{1-(-1/2)^2} + \sqrt{1-0^2} + \sqrt{1-(1/2)^2} + \sqrt{1-1^2}\right)\cdot\frac{1}{2} \\
&= \left(\frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2}\right)\cdot\frac{1}{2} \\
&= \frac{\sqrt{3}+1}{2}
\end{aligned}$$

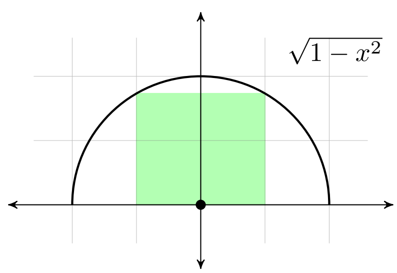
The upper sum requires a little bit of extra work. We have to figure out where f is largest on *each subinterval $[x_i, x_{i+1}]$, and this will vary depending on i. Let us look at a picture:*



Thus, for example, on the interval $[x_0, x_1] = [-1, -\frac{1}{2}]$, it is clear that f is largest when $x = -\frac{1}{2}$, sowe must choose $t_0 = -\frac{1}{2}$. Proceeding analogously, we get $t_1 = 0$, $t_2 = 0$, and $t_3 = \frac{1}{2}$, and so

$$\begin{aligned}
U &= \sum_{i=0}^{n-1} f(t_i)\Delta x \\
&= f(-\frac{1}{2})\cdot\frac{1}{2} + f(0)\cdot\frac{1}{2} + f(0)\cdot\frac{1}{2} + f(\frac{1}{2})\cdot\frac{1}{2} \\
&= \left(\sqrt{1-(-1/2)^2} + \sqrt{1-0^2} + \sqrt{1-0^2} + \sqrt{1-(1/2)^2}\right)\cdot\frac{1}{2} \\
&= \left(\frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + 0\right)\cdot\frac{1}{2} \\
&= \frac{\sqrt{3}+2}{2}
\end{aligned}$$

Similarly, to compute the lower Riemann sum, we see from the picture



that, for example on $[x_0, x_1] = [-1, -\frac{1}{2}]$, f is smallest at $x = -1$, so we will need $t_0 = -1$, and proceeding analogously we get that $t_1 = -\frac{1}{2}$, $t_2 = \frac{1}{2}$ and $t_3 = 1$. Thus,

$$\begin{aligned}
L &= \sum_{i=0}^{n-1} f(t_i)\Delta x \\
&= f(-1)\cdot\frac{1}{2} + f(-\frac{1}{2})\cdot\frac{1}{2} + f(\frac{1}{2})\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} \\
&= \left(\sqrt{1-(-1)^2} + \sqrt{1-(-1/2)^2} + \sqrt{1-(1/2)^2} + \sqrt{1-1^2}\right)\cdot\frac{1}{2} \\
&= \left(0 + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + 0\right)\cdot\frac{1}{2} \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

As a final observation, we note that $L \leq S_L = S_R \leq U$. ■

The next step is, of course, refining the partition of $[a, b]$ and summing over more and more rectangles, thus, hopefully, getting a more accurate approximation to the 'actual' area. A function f is said to be **Riemann integrable on** $[a, b]$ if, no matter how the choice of partitions P of $[a, b]$ is made, the limit of these Riemann sums as we add more and more rectangles,

$$\lim_{n\to\infty} S_P \sum_{i=1}^{n} f(t_i) \cdot (x_{i+1} - x_i)$$

exists as a real number. It is a fact, proven later in undergraduate analysis, that the choice of partition doesn't matter, so you may choose, if you want, the left Riemann sum every time. In this case, if the limit exists, f is said to be **Riemann integrable** on $[a, b]$ and the value of this limit is called the **Riemann integral** or **definite Riemann integral** of f on $[a, b]$, and this limit is denoted

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} S_P = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$$

If we choose $\Delta x = x_{i+1} - x_i$ to be constant and equal to $\frac{b-a}{n}$, then we may write

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\Delta x$$

2 Theorems

Theorem 2.1 (Limit Laws) *Let k be a constant real number. Then,*

1. $\lim_{x\to c}(kf(x)) = k\lim_{x\to c} f(x)$
2. $\lim_{x\to c}\big(f(x)+g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
3. $\lim_{x\to c}\big(f(x)g(x)\big) = \big(\lim_{x\to c} f(x)\big)\big(\lim_{x\to c} g(x)\big)$
4. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ *if* $g(x)\neq 0$
5. $\lim_{x\to c} k = k$ *(we think of k as a constant function)*
6. $\lim_{x\to c} x = c$ *(we think of x as the function $f(x)=x$)* ■

Theorem 2.2 (Continuity Laws) *Let k be a constant real number and suppose f and g are continuous on an interval $[a,b]$ (or an open or half-open interval, it doesn't matter). Then, the following functions are continuous:*

1. $kf(x)$
2. $f(x)+g(x)$
3. $f(x)g(x)$
4. $\dfrac{f(x)}{g(x)}$ *if* $g(x)\neq 0$
5. $(f\circ g)(x)$ ■

Theorem 2.3 (Intermediate Value Theorem) *If f is continuous on a closed and bounded interval $[a,b]$ and k is a y-value between $f(a)$ and $f(b)$ (whether $f(a)\leq k\leq f(b)$ or $f(b)\leq k\leq f(a)$), then there is at least one number c between a and b such that $f(c)=k$.* ■

Remark 2.4 *This means that the entire range of y-values between $f(a)$ and $f(b)$ is hit by f on the interval $[a,b]$, possibly more than once.* ■

Example 2.5 *Since $f(x)=\sqrt{x}$ is continuous on $[4,100]$, and since $f(4)=2$ and $f(100)=10$, all numbers between 2 and 10 are hit. For example 5: there is a number, namely $c=25$, in between 4 and 100, such that $f(c)=5$.* ■

Example 2.6 *Consider the polynomial $p(x)=5x^3+\pi x+1$. It has at least one root between -1 and 0, because $p(-1)=-4-\pi$ and $p(0)=1$, and since 0 lies in between these two y values and $p(x)$ is continuous on $[-1,0]$, we know there is a number c between -1 and 0 such that $p(c)=0$. That's our root.* ■

Example 2.7 *Consider $f(x) = 1/x$. Then $f(-1) = -1$ and $f(1) = 1$, but the Intermediate Value Theorem doesn't apply to give a c between -1 and 1 such that $f(c) = 0$, because f is not continuous on $[-1, 1]$. Indeed, $f(x)$ never equals 0.* ■

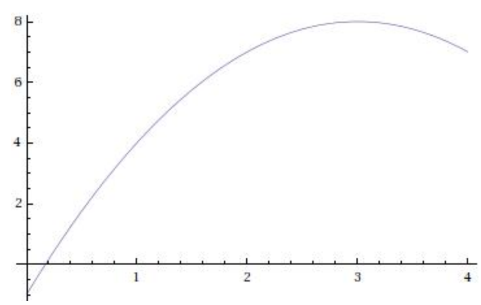
Theorem 2.8 (Extreme Value Theorem) *If f is a continuous function on a closed and bounded interval $[a, b]$, then f attains a (global) minimum and a (global) maximum on that interval, that is $\max_{a \le x \le b} f(x)$ and $\min_{a \le x \le b} f(x)$ exist as real numbers. This means there are points x_1 and x_2 in the interval $[a, b]$ such that*

$$f(x_1) = \max_{a \le x \le b} f(x) \qquad \text{and} \qquad f(x_2) = \min_{a \le x \le b} f(x)$$ ■

Example 2.9 *The parabola $f(x) = -x^2 + 6x - 1$, being a polynomial, is continuous everywhere, so for example on $[0, 4]$ it must achieve its minimum and its maximum. In fact,*

$$f(0) = \min_{0 \le x \le 4} f(x) = -1 \qquad \text{and} \qquad f(3) = \max_{0 \le x \le 4} f(x) = 8$$

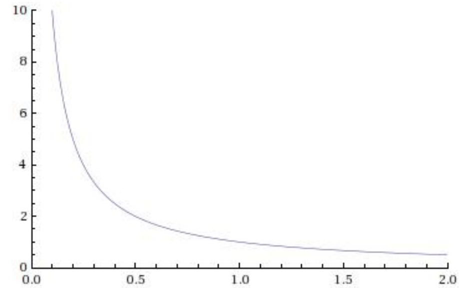
as can be seen from the graph:



Of course, we don't need the graph, because (using the first and second derivative tests) we can actually find the maximum and minimum analytically as follows:

1. *First, find all critical points, which means ompute the derivative, set it equal to zero and solve for x: $f'(x) = -2x + 6 = 0$, so $x = 3$ is a critical point.*
2. *Use, e.g. the second derivative test to classify it: $f''(3) = -2 < 0$, so it's a local max.*
3. *Check the value of f at the critical point (which lies in the interval $[0, 4]$) against the value of f at the endpoints 0 and 4: $f(0) = -1$, $f(3) = 8$ and $f(4) = 7$, so it looks like 0 is the global min and 3 is the global max here, i.e. $f(0) = \min_{0 \le x \le 4} f(x) = -1$ and $f(3) = \max_{0 \le x \le 4} f(x) = 8$.* ■

Example 2.10 *The function $f(x) = \frac{1}{x}$ is continuous on $(-\infty, 0) \cup (0, \infty)$, but on $(0, 2)$ the Extreme Value Theorem doesn't apply, because $(0, 1)$ is not a closed interval. As it turns out, $f(x)$ has neither a (local or global) max or min on $(0, 1)$ (it just gets bigger and bigger without bound as x approaches 0 and it gets closer and closer to $1/2$ as x approaches 2, but it never reaches $1/2$, and so it never reaches a smallest value.*



Theorem 2.11 (Differentiation Rules) *If two functions f and g are differentiable at $x = a$ and k is a constant, then the functions c (the constant function), $cf(x)$, $f(x) + g(x)$, $f(x)g(x)$, $f(x)/g(x)$ (when $g(x) \neq 0$), and $(f \circ g)(a)$ (when f is differentiable at $b = g(a)$) are all differentiable at $x = a$, and moreover satisfy the following formulas:*

(1) $c' = 0$

(2) $(cf)'(a) = cf'(a)$ (*constant factor rule*) } (*linearity*)

(3) $(f+g)'(a) = f'(a) + g'(a)$ (*sum rule*) } (*linearity*)

(4) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$ (*product rule*)

(5) $\left(\dfrac{f}{g}\right)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$ (*quotient rule*)

(6) $(f \circ g)'(a) = f'\big(g(a)\big) \cdot f'(a)$ (*chain rule*)

■

Theorem 2.12 (Differentiability Implies Continuity) *If f is differentiable at a point $x = a$, then it is continuous at $x = a$. Consequently, if f is differentiable on an open interval (a, b), then it is continuous on that interval.* ■

Remark 2.13 *Of course the converse is not true. You can have a continuous function which is not differentiable. For example $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there (the left and right limits of the difference quotient as $h \to 0$ don't agree).* ■

Theorem 2.14 (Mean Value Theorem) *If f is continuous on $[a, b]$ and differentiable on (a, b), then there is some c between a and b such that*

$$f(b) - f(a) = f'(c)(b - a)$$

■

Example 2.15 *The Mean Value Theorem is used for many things, but the simplest of its uses is in **proving inequalities**. For example, let us prove the inequality*

$$\sin x \leq x$$

on $[0, \infty)$. The trick is to define a new function, $h(x)$, as the difference of the two functions under consideration, that is let

$$h(x) = x - \sin x$$

Then we need only show that $h(x) \geq 0$ to prove our assertion. Now, since for any $x > 0$ we know that h is continous on $[0, x]$ and differentiable on $(0, x)$, the MVT applies to give the existence of a c between 0 and x such that

$$h(x) - h(0) = h'(c)(x - 0)$$

Now, $h(0) = 0 - \sin 0 = 0$, and moreover $h'(x) = 1 - \cos x \geq 0$, which, along with the fact that $x > 0$, implies $h'(x)(x - 0) \geq 0$, so we have

$$x - \sin x = h(x) \geq 0$$

on $[0, \infty)$, which is what we set out to prove. ■

Corollary 2.16 (Constant Function Theorem) *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then $f'(x) = 0$ on all of (a, b) if and only if f is constant on the interval.*

Corollary 2.17 *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = g'(x)$ on (a, b), then f and g differ by a constant, i.e.*

$$f(x) = g(x) + c$$

for some constant c.

Remark 2.18 *This is important for integrals and anti-derivatives. Namely, all anti-derivatives differ by a constant, and so the most general antiderivative of a function is written $\int f(x)dx = F(x) + C$ where $F(x)$ is some specific antiderivative and C is an arbitrary constant, that constant showing up in the last corollary.* ■

Corollary 2.19 *Let f be differentiable on an open interval (a, b).*

(1) *If $f'(x) \geq 0$ on (a, b), then f is increasing on (a, b).*
(2) *If $f'(x) \leq 0$ on (a, b), then f is decreasing on (a, b).*

Moreover, if the inequalities are strict, then f is strictly increasing or strictly decreasing, respectively. ■

Corollary 2.20 (First Derivative Test) *Let f be differentiable on an open interval containing a critical point x_0 of f, i.e. one for which $f'(x_0) = 0$.*

(1) *If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, then x_0 is a local maximum of f.*
(2) *If $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$, then x_0 is a local minimum of f.*

Corollary 2.21 (Second Derivative Test) *Let f be twice differentiable on an open interval containing a critical point x_0 of f, i.e. one for which $f'(x_0) = 0$.*

(1) *If $f''(x_0) > 0$, then x_0 is a local minimum of f.*

(2) *If $f''(x_0) < 0$, then x_0 is a local maximum of f.*

(3) *If $f''(x_0) = 0$, then no conclusion can be drawn. x_0 may be a local minimum or a local maximum or a point of inflection.* ■

Theorem 2.22 (Inverse Fuction Theorem) *Let f be continuous and invertible on $[a, b]$ and differentiable at some point x_0 in $[a, b]$. If $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$ and if we let $y_0 = f(x_0)$, then*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$ ■

Theorem 2.23 (L'Hôspital's Rule) *Let f and g be differentiable on (a, b) and let $g'(x) \neq 0$ on (a, b). We allow $a = -\infty$ and $b = \infty$. If for some x_0 in $[a, b]$ we have $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ or $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm\infty$, and if the limit*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

exists as a real number or as $\pm\infty$, then the limit $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists as a real number or $\pm\infty$, and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$ ■

Theorem 2.24 *If f and g and continuous on $[a, b]$ and k is a constant, then (linearity)*

$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

$$\int_a^b k f(x)\, dx = k \int_a^b f(x)\, dx$$

and if $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

As a consequence, if $m \leq f(x) \leq M$ on $[a, b]$, then

$$m(b - a) \leq \int_a^b f(x)\, dx \leq M(b - a)$$ ■

Theorem 2.25 *Let f be Riemann continuous on $[a,b]$. Then for all c in $[a,b]$ we have*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$ ■

Theorem 2.26 *f f is continous on $[a,b]$, then*

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Remark 2.27 *This is more a matter of convention than a theorem.* ■

Theorem 2.28 (Average/Mean Value Theorem for Integrals) *If f is continuous on $[a,b]$, then there is a number c in $[a,b]$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 2.29 (Fundamental Theorem of Calculus I) *If f is continuous on $[a,b]$ and differentiable on (a,b), and if f' is continuous on (a,b), then*

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$ ■

Theorem 2.30 (Fundamental Theorem of Calculus II) *If f is continuous on $[a,b]$, then the function $F : [a,b] \to \mathbb{R}$ given by*

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b), with derivative given by

$$F'(x) = f(x)$$ ■

Remark 2.31 *This theorem, FTC II, gives the existence of antiderivatives for any Riemann integrable function on $[a,b]$. However, this result is only of theoretical interest, as the form of the antiderivative, $F(x) = \int_a^x f(t)\,dt$, isn't really computationally helpful. However, it is a useful for familiarizing yourself with the notation and definitions. For example, if we want to take the x-derivative of $F(\sin x)$, this is straightforward, just apply the* ***chain rule****: letting $y = \sin x$,*

$$\frac{d}{dx}F(\sin x) = \frac{dF}{dy}\frac{dy}{dx} = f(x)\cos x$$

But it's slightly harder to see in the form $\frac{d}{dx}\int_a^{\sin x} f(t)\,dt$. However, the two expressions are the same!

$$\frac{d}{dx}\int_a^{\sin x} f(t)\,dt = \frac{dF}{dy}\frac{dy}{dx} = f(x)\cos x$$ ■

Theorem 2.32 (Integration by Parts) *If f and g are continuous on $[a,b]$ and differentiable on (a,b), and if f' and g' are Riemann integrable on $[a,b]$, then*

$$\int_a^b f(x)g'(x)\,dx = \Big[f(x)g(x)\Big]_{x=a}^{b} - \int_a^b f'(x)g(x)\,dx \tag{2.1}$$

■

Theorem 2.33 (Change of Variable) *If $f : [a,b] \to [\alpha,\beta]$ is continously differentiable on $[a,b]$ and g is continuous on $[\alpha,\beta]$, then*

$$\int_a^b g\big(f(x)\big)f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy \tag{2.2}$$

■