1 Points and Vectors

1.1 Definitions

We denote \( n \)-dimensional Euclidean space by

\[
\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}
= \{(x_1, \ldots, x_n) \mid x_i \text{ is a real number}\}
\] (1.1)

We will typically use capital letters \( P \) for the elements of \( \mathbb{R}^n \), the \( n \)-tuples \((x_1, \ldots, x_n)\), which we call points,

\[ P = (x_1, \ldots, x_n) \] (1.2)

We may also think of the elements of \( \mathbb{R}^n \) as vectors, however, for example when working with things like velocities/forces/etc. ('vector quantities', typically residing in phase space rather than configuration space, in physics lingo). There is a certain notation which is found in both physics and math books, that emphasises the vector part of the elements of \( \mathbb{R}^n \), and it is the arrow and angle bracket notations for the \( n \)-tuples:

\[ \vec{x} \text{ or } \mathbf{x} = \langle x_1, \ldots, x_n \rangle \] (1.3)

**Example 1.1** Our two main examples are the Euclidean plane \( \mathbb{R}^2 \) and Euclidean three-dimensional space \( \mathbb{R}^3 \). We usually denote \( x_1 \) by \( x \), \( x_2 \) by \( y \), and \( x_3 \) by \( z \), so that \( P = (x, y) \) or \((x, y, z)\), as the case may be.

We visualize points and vectors differently. For example, in \( \mathbb{R}^2 \) the point \( P = (-1, 3) \) is pictured as a dot, and the vector \( \vec{x} = (-1, 3) \) is pictured as an arrow from the origin to \( P \):

![Diagram showing a point and a vector in 2D space.](image-url)
1.2 Algebraic Properties of Vectors

The essential difference between points and vectors, mathematically, is that points don’t possess any algebraic properties, whereas vectors do. The key algebraic properties of vectors are addition, scalar multiplication, and the dot and cross products:

1. We can add two vectors.
2. We can scale any vector (multiply it by a real number).
3. Rules (1) and (2) are subject to certain rules (associativity, commutativity and distributivity rules) which will make them “nice” to work with and give them geometric content.\footnote{In the case of Euclidean \( \mathbb{R}^n \) the rules for addition and scalar multiplication follow from directly from the definitions of addition and scalar multiplication themselves, but in abstract linear algebra they form the basis for the definition of (abstract) vector space.}
4. We can “multiply” two vectors, in different ways. The dot product of two vectors results in a real number.
5. The cross product results in another vector.

1.2.1 Vector Addition

We define addition of vectors in \( \mathbb{R}^n \) componentwise,

\[
\vec{x} + \vec{y} = \langle x_1, x_2, \ldots, x_n \rangle + \langle y_1, y_2, \ldots, y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n \rangle
\]

Example 1.2 Let us see what this means in \( \mathbb{R}^2 \). Take, say, \( \vec{x} = \langle 1, 2 \rangle \) and \( \vec{y} = \langle -2, 1 \rangle \). Then \( \vec{x} + \vec{y} = \langle 1 - 2, 2 + 1 \rangle = \langle -1, 3 \rangle \).

Thus we see that to reach \( \vec{x} + \vec{y} \), we may first go to \( \vec{x} \), then go in the direction of \( \vec{y} \) to get to \( \vec{x} + \vec{y} \), or else we may to to \( \vec{y} \) first and then go in the direction of \( \vec{x} \). This shows geometrically the algebraic rule called \textbf{commutativity}, \( \vec{x} + \vec{y} = \vec{y} + \vec{x} \).

As a direct consequence of our definition of vector addition see that we have \textbf{commutativity}
of addition:
\[
\vec{x} + \vec{y} = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) = (y_1 + x_1, \ldots, y_n + x_n) = \vec{y} + \vec{x}
\] (1.5)

Another obvious fact about our definition of addition in \( \mathbb{R}^n \) is that it is **associative**. We can add two vectors first, then a third, but we could just as well have added the second and third first, then the first to the result:
\[
(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})
\] (1.6)

which again follows from the same associativity holding in each component, \((x_i + y_i) + z_i = x_i + (y_i + z_i)\).

It is also clear that the **zero** vector,
\[
\vec{0} = (0, \ldots, 0)
\] (1.7)

satisfies
\[
\vec{0} + \vec{x} = \vec{x} + \vec{0} = \vec{x}
\] (1.8)

for all \( \vec{x} \) in \( \mathbb{R}^n \). Moreover, the **negative** or a vector \( \vec{x} \) in \( \mathbb{R}^n \), defined by
\[
-\vec{x} = (-x_1, \ldots, -x_n)
\] (1.9)

satisfies
\[
(-\vec{x}) + \vec{x} = \vec{x} + (-\vec{x}) = \vec{0}
\]

which we may more compactly write \(-\vec{x} + \vec{x} = \vec{x} - \vec{x} = \vec{0}\). That is, we may use negative vectors to define **subtraction** of vectors in \( \mathbb{R}^n \), namely by addition of negatives:
\[
\vec{x} - \vec{y} = \vec{x} + (-\vec{y})
\] (1.10)

Geometrically, the negative \(-\vec{x}\) of a vector is the reflection of \(\vec{x}\) through the origin:
1.2.2 Scalar Multiplication

Let us now define scalar multiplication of vectors in $\mathbb{R}^n$, meaning multiplication of a vector $\vec{x}$ by a real number $a$. As with addition, this is defined componentwise:

$$a\vec{x} = a(x_1, \ldots, x_n) = (ax_1, \ldots, ax_n) \quad (1.11)$$

It is clear that

$$1\vec{x} = \vec{x} \quad (1.12)$$
$$0\vec{x} = \vec{0} \quad (1.13)$$

Moreover, we have associativity of scalar multiplication,

$$a(b\vec{x}) = (ab)\vec{x} \quad (1.14)$$

and distributivity of scalar multiplication over addition:

$$a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y} \quad (1.15)$$
$$\langle a + b \rangle \vec{x} = a\vec{x} + b\vec{x} \quad (1.16)$$

for all real numbers $a$ and $b$ and all elements $\vec{x}$ and $\vec{y}$ of $\mathbb{R}^n$.

The geometric content of scalar multiplication may be seen in the following example:

**Example 1.3** Take, say, the vector $\vec{x} = \langle 1, 2 \rangle$ and the real number $a = 2$. Then, algebraically,

$$a\vec{x} = 2\langle 1, 2 \rangle = \langle 2 \cdot 1, 2 \cdot 2 \rangle = \langle 2, 4 \rangle$$

which, geometrically means this:

Thus geometrically scalar multiplication has the effect of scaling the length of the vector $\vec{x}$.
1.2.3 Coordinate Vectors and Vector Decomposition

We can use addition and scalar multiplication of vectors to decompose a given vector \( \vec{x} = \langle x_1, \ldots, x_n \rangle \) into its components \( x_i \):

\[
\vec{x} = (x_1, \ldots, x_n) = (x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_n)
\]

If we define the coordinate basis vectors

\[
e_1 = \langle 1, 0, \ldots, 0 \rangle, \quad e_2 = \langle 0, 1, 0, \ldots, 0 \rangle, \quad \ldots, \quad e_n = \langle 0, \ldots, 0, 1 \rangle
\]

then we can nicely rewrite equation (1.17) as

\[
\vec{x} = (x_1, \ldots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n
\]

**Example 1.4** When \( n = 2 \), we have special notation, which is more commonly found in physics texts:

\[
\vec{i} = i = e_1 = \langle 1, 0 \rangle
\]

\[
\vec{j} = j = e_2 = \langle 0, 1 \rangle
\]

When \( n = 3 \), we also write:

\[
\vec{i} = i = e_1 = \langle 1, 0, 0 \rangle
\]

\[
\vec{j} = j = e_2 = \langle 0, 1, 0 \rangle
\]

\[
\vec{k} = k = e_3 = \langle 0, 0, 1 \rangle
\]

For example,

\[
\langle 1, 3, -2 \rangle = 1\vec{i} + 3\vec{j} - 2\vec{k}
\]

is decomposed into its components. As another example,

\[
\langle 4, 3 \rangle = 4\vec{i} + 3\vec{j}
\]

and this can be pictured as follows:

![Diagram](image-url)
1.2.4 The Relationship Between Points and Vectors: the Displacement Vector

Given two points \( P = (x_1, \ldots, x_n) \) and \( Q = (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \), the displacement vector from \( P \) to \( Q \) is defined as

\[
\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (y_1 - x_1, \ldots, y_n - x_n)
\]  

(1.23)

We picture the displacement vector \( \overrightarrow{PQ} \) as emanating from the point \( P \) and ending in an arrow at the point \( Q \) (even though it always, strictly speaking, emanates from the origin to its endpoint).

**Example 1.5** Let us look at the case of \( \mathbb{R}^2 \). Suppose \( P = (1, 2) \) and \( Q = (5, 3) \). Then,

\[
\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (5, 3) - (1, 2) = (5 - 1, 3 - 2) = (4, 1)
\]

(1.24)

and the picture is this:

![Diagram](image)

**Remark 1.6** In fact, \( \overrightarrow{PQ} \) should be pictured as emanating from the origin, but we want to think of \( \overrightarrow{PQ} \) as emanating from \( P \). We should, then, if we were being rigorous, think of \( \overrightarrow{PQ} \) as lying in a copy of \( \mathbb{R}^n \) sitting above our position space \( \mathbb{R}^n \) at the point \( P \), that is we should think of \( \overrightarrow{PQ} \) as lying in the set \( \{ P \} \times \mathbb{R}^n = \{ (P, \vec{x}) \mid \vec{x} = (x_1, \ldots, x_n) \} \), and thus

\[
\overrightarrow{PQ} = (P, \overrightarrow{OQ} - \overrightarrow{OP})
\]

For example, if \( P = (1, 2) \) and \( Q = (5, 3) \), then

\[
\overrightarrow{PQ} = \left( (1, 2), \langle 4, 1 \rangle \right)
\]

We will not nit-pick here, and we will simply conflate points and vectors in the strict sense, but we will picture vectors as emanating from points in the underlying position space. □

Now suppose we are considering not two points \( P \) and \( Q \), but two vectors \( \vec{v} \) and \( \vec{w} \). Then we can consider the displacement vector from \( \vec{v} \) to \( \vec{w} \). This is, in fact \( \vec{w} - \vec{v} \), which is simply
due to the fact that \( \vec{w} = \vec{v} + (\vec{w} - \vec{v}) \):

This of course harmonizes with our previous definition, \( \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \). We subtract our starting position vector from our ending position vector in both cases.

1.2.5 The Dot Product

The dot product of two vectors \( \vec{x} = \langle x_1, x_2, \ldots, x_n \rangle \) and \( \vec{y} = \langle y_1, y_2, \ldots, y_n \rangle \) in \( \mathbb{R}^n \) is defined by

\[
\vec{x} \cdot \vec{y} = \langle x_1, x_2, \ldots, x_n \rangle \cdot \langle y_1, y_2, \ldots, y_n \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n
\]

or, more concisely, using summation notation,

\[
\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i
\]

**Example 1.7** Consider the vectors \( \vec{x} = (1, 2, -5) \) and \( \vec{y} = (-3, 2, 4) \) in \( \mathbb{R}^3 \). Their dot product is

\[
\vec{x} \cdot \vec{y} = (1, 2, -5) \cdot (-3, 2, 4) = 1 \cdot (-3) + 2 \cdot 2 + (-5) \cdot 4 = -19
\]

The length (or magnitude or norm) of a vector \( \vec{x} \) in \( \mathbb{R}^n \) will be defined as the square root of the dot product of \( \vec{x} \) with itself:

\[
|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}}
\]

Thus,

\[
|\vec{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}
\]

and therefore

\[
|\vec{x}|^2 = \sum_{i=1}^{n} x_i^2
\]
This relates to the formula for the distance between two points \( P = (x_1, \ldots, x_n) \) and \( Q = (y_1, \ldots, y_n) \):

\[
d(P, Q) = d\left((x_1, \ldots, x_n), (y_1, \ldots, y_n)\right) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}
\]

(1.28)

In the cases \( n = 2 \) and \( n = 3 \), this is the Pythagorean theorem. For \( n > 3 \) it’s simply a definition which in a way takes the Pythagorean theorem as an axiom.

What is the relationship between the distance formula and the length of a vector? It is that the length of the displacement vector \( \overrightarrow{PQ} \) is precisely the distance between \( P \) and \( Q \):

\[
|\overrightarrow{PQ}| = \sqrt{\overrightarrow{PQ} \cdot \overrightarrow{PQ}} = \sqrt{(\vec{Q} - \vec{P}) \cdot (\vec{Q} - \vec{P})} = \sqrt{(y_1 - x_1, \ldots, y_n - x_n) \cdot (y_1 - x_1, \ldots, y_n - x_n)} = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} = d(P, Q)
\]

The proofs of the following properties of the dot product are analogous to the above calculation showing that \( |\overrightarrow{PQ}| = d(P, Q) \), and we leave them as an easy exercise.

**Example 1.8** Let \( P = (1, 2) \) and \( Q = (5, 3) \) be points in the plane \( \mathbb{R}^2 \). Find the magnitude of the displacement vector from \( P \) to \( Q \).

Solution: Let \( \vec{x} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (4, 1) \), then

\[
|\vec{x}| = |\langle 4, 1 \rangle| = \sqrt{4^2 + 1^2} = \sqrt{17}
\]

**Proposition 1.9 (Algebraic Properties of the Dot Product)** Let \( \vec{x}, \vec{y}, \vec{z} \) be vectors in \( \mathbb{R}^n \) and let \( c \) be a real number. Then,

1. \( \vec{x} \cdot \vec{x} = |\vec{x}|^2 \)
2. \( |c\vec{x}| = |c||\vec{x}| \)
3. \( \vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x} \) (commutativity)
4. \( \vec{x} \cdot (\vec{y} + \vec{z}) = \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z} \) (distributivity over addition)
5. \( (c\vec{x}) \cdot \vec{y} = c(\vec{x} \cdot \vec{y}) = \vec{x} \cdot (c\vec{y}) \) (associativity and commutativity of scalar multiplication and dot multiplication)
6. \( \vec{0} \cdot \vec{x} = 0 \)
In two and three dimensions, the dot product has a very geometric interpretation:

**Proposition 1.10** Let \( \vec{x} = (x_1, y_1, z_1) \) and let \( \vec{y} = (x_2, y_2, z_2) \) be two vectors in \( \mathbb{R}^3 \), and let \( \theta \) be the angle between them. Then,

\[
\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \theta
\]  

(1.29)

**Proof:** Consider the triangle formed by the vectors \( \vec{x}, \vec{y} \) and \( \vec{y} - \vec{x} \).

By the Law of Cosines

\[
|\vec{y} - \vec{x}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}| |\vec{y}| \cos \theta
\]

Using the fact that \( |\vec{x}|^2 = \vec{x} \cdot \vec{x} \), we can rewrite the above as

\[
(\vec{y} - \vec{x}) \cdot (\vec{y} - \vec{x}) = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} - 2|\vec{x}| |\vec{y}| \cos \theta
\]

Distributing on the left and simplifying, we get

\[
\vec{y} \cdot \vec{y} - 2\vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{x} = \vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} - 2|\vec{x}| |\vec{y}| \cos \theta
\]

That is,

\[
-2\vec{x} \cdot \vec{y} = -2|\vec{x}| |\vec{y}| \cos \theta
\]

\[
\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \theta
\]

\[
\Box
\]

### 1.2.6 Cross Product

*Note that this section requires knowledge of determinants, which is to be found below, in the section on matrices.*

The cross product is a vector product, meaning that multiplying two vectors this way results in another vector (this is why the dot product is sometimes called the scalar product, to distinguish it from this vector product). The cross product is only defined in 3 dimensions, i.e. only on \( \mathbb{R}^3 \).\(^{2}\)

\(^2\)It generalizes to other dimensions only once we switch to the wedge product (or exterior product) in multilinear algebra. Moving past the algebra we are led to a differential type of wedge product in the apparatus of differential forms. For further reading on this, see Knapp [2] and Gallier [1] for the algebra, and Munkres [3] for the calculus side.
Let \( \vec{u} = \langle a, b, c \rangle \) and \( \vec{v} = \langle d, e, f \rangle \) be two vectors in \( \mathbb{R}^3 \). We define their cross product to be the vector gotten by computing the following determinant:

\[
\vec{u} \times \vec{v} = \langle a, b, c \rangle \times \langle d, e, f \rangle = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
a & b & c \\
d & e & f
\end{vmatrix}
= \vec{i} \begin{vmatrix} b & c \\ e & f \end{vmatrix} - \vec{j} \begin{vmatrix} a & c \\ d & f \end{vmatrix} + \vec{k} \begin{vmatrix} a & b \\ d & e \end{vmatrix}
\]

(1.30)

Carrying out this computation to its awful end, we get

\[
\vec{u} \times \vec{v} = (bf - ce)\vec{i} - (af - cd)\vec{j} + (ae - bd)\vec{k} = \langle bf - ce, cd - af, ae - bd \rangle
\]

But it is easier to remember the equation (1.30) in terms of determinants and just perform the rest of the computation by hand in particular cases.

**Example 1.11** Let \( \vec{u} = \langle 1, 2, -2 \rangle \) and \( \vec{v} = \langle -8, 5, 4 \rangle \). Then,

\[
\vec{u} \times \vec{v} = \langle 1, 2, -2 \rangle \times \langle -8, 5, 4 \rangle
= \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 2 & -2 \\
-8 & 5 & 4
\end{vmatrix}
= \vec{i} \begin{vmatrix} 2 & -2 \\ 5 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & -2 \\ -8 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 2 \\ -8 & 5 \end{vmatrix}
= (8 + 10)\vec{i} - (4 - 16)\vec{j} + (5 + 16)\vec{k}
= 18\vec{i} + 12\vec{j} + 21\vec{k}
\text{or} \quad \langle 18, 12, 21 \rangle
\]

**Example 1.12** Let us compute \( \vec{i} \times \vec{j} \):

\[
\vec{i} \times \vec{j} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 0 & 0 \\
0 & 1 & 0
\end{vmatrix} = \vec{i} \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 1\vec{k} = \vec{k}
\]

By similar calculations, which we leave to you, we also have the relations \( \vec{j} \times \vec{k} = \vec{i} \) and \( \vec{k} \times \vec{i} = \vec{j} \):

\[
\vec{i} \times \vec{j} = \vec{k}
\vec{j} \times \vec{k} = \vec{i}
\vec{k} \times \vec{i} = \vec{j}
\]

(1.31)

Note that the vectors \( \vec{i}, \vec{j}, \vec{k} \) are cyclically permuted:

\[
\vec{i} \times \vec{j} = \vec{k}
\vec{j} \times \vec{k} = \vec{i}
\vec{k} \times \vec{i} = \vec{j}
\]

(1.32)

\[
\text{in slots of the equation} \quad \vec{a} \times \vec{b} = \vec{c}
\]
Proposition 1.13 (Algebraic Properties of the Cross Product) Let \( \vec{x}, \vec{y}, \vec{z} \) be vectors in \( \mathbb{R}^3 \) and let \( c \) be a real number. Then,

1. \( \vec{x} \times \vec{y} = -\vec{y} \times \vec{x} \) (anti-commutativity)
2. \( \vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z} \) (distributivity over addition)
3. \( (c\vec{x}) \times \vec{y} = c(\vec{x} \times \vec{y}) = \vec{x} \times (c\vec{y}) \) (associativity of scalar multiplication and cross multiplication)

Proof: Let us prove (1), and leave the rest as easy exercises. Let \( \vec{x} = \langle a, b, c \rangle \) and \( \vec{y} = \langle d, e, f \rangle \). Then,

\[
\vec{y} \times \vec{x} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
d & e & f \\
a & b & c
\end{vmatrix} = \begin{vmatrix}
e & f \\
d & a & c \\
a & b & c
\end{vmatrix} - \begin{vmatrix}
d & f \\
e & a & c \\
b & a & c
\end{vmatrix} = (ce - bf)\vec{i} - (af - cd)\vec{j} + (bd - ae)\vec{k}
\]

However, according to the calculation (1.30) above,

\[
\vec{x} \times \vec{y} = (bf - ce)\vec{i} - (af - cd)\vec{j} + (ae - bd)\vec{k} = -\vec{y} \times \vec{x}
\]

Proposition 1.14 For any vectors \( \vec{u}, \vec{v} \) in \( \mathbb{R}^3 \) we have the identity

\[
|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta
\]  

(1.33)

and consequently

\[
\vec{u} \times \vec{v} = (|\vec{u}||\vec{v}| \sin \theta)\vec{n}
\]  

(1.34)

where \( \theta \) is the angle between the vectors and \( \vec{n} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} \) is the unit vector in the direction of \( \vec{u} \times \vec{v} \), determined by the right-hand rule, illustrated in the following diagram,\(^3\)

![Diagram of the right-hand rule](http://en.wikipedia.org/wiki/Cross_product)

Proof: Let \( \vec{u} = \langle a, b, c \rangle \) and \( \vec{v} = \langle d, e, f \rangle \). By the calculation (1.30) we have that \( \vec{u} \times \vec{v} = \)

\langle bf - ce, cd - af, ae - bd \rangle, and consequently

\[ |\vec{u}|^2 |\vec{v}|^2 \sin^2 \theta = |\vec{u}|^2 |\vec{v}|^2 (1 - \cos^2 \theta) \]
\[ = |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta \]
\[ = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \]
\[ = (a^2 + b^2 + c^2)(d^2 + e^2 + f^2) - (ad + be + cf)^2 \]
\[ = (a^2 d^2 + b^2 e^2 + c^2 f^2 + b^2 f^2 + c^2 e^2 + c^2 d^2 + a^2 f^2 + a^2 e^2 + b^2 d^2) \]
\[ - (a^2 d^2 + b^2 e^2 + c^2 f^2 + 2bcef + 2acdf + 2abd) \]
\[ = (b^2 f^2 - 2bcef + c^2 e^2) + (c^2 d^2 - 2acdf + a^2 f^2) + (a^2 e^2 - 2abde + b^2 d^2) \]
\[ + (a^2 d^2 + b^2 e^2 + c^2 f^2) - (a^2 d^2 + b^2 e^2 + c^2 f^2) \]
\[ = (bf - ce)^2 + (cd - af)^2 + (ae - bd)^2 \]
\[ = \langle bf - ce, cd - af, ae - bd \rangle \cdot (bf - ce, cd - af, ae - bd) \]
\[ = |\vec{u} \times \vec{v}|^2 \]

Taking the square root gives the result. ■
1.2.7 Interaction Between the Dot and Cross Products

**Proposition 1.15** Let \( \vec{a} = \langle a_1, a_2, a_3 \rangle \), \( \vec{b} = \langle b_1, b_2, b_3 \rangle \), \( \vec{c} = \langle c_1, c_2, c_3 \rangle \) be vectors in \( \mathbb{R}^3 \). Then,

\[
\begin{align*}
(1) \quad \vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \\
(\vec{a} \times \vec{b}) \times \vec{c} &= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a} & \text{(vector triple product)}
\end{align*}
\]

\[
(2) \quad (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{c} \times \vec{a}) \cdot \vec{b} = (\vec{b} \times \vec{c}) \cdot \vec{a} = \begin{vmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix} = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a}) & \text{(scalar triple product)}
\]

\[
(3) \quad (\vec{a} \times \vec{b}) \cdot \vec{c} + (\vec{c} \times \vec{a}) \times \vec{b} + (\vec{b} \times \vec{c}) \times \vec{a} = \vec{0} & \text{(Jacobi identity)}
\]

**Proof:** (1) This is a direct computation:

\[
\begin{align*}
\vec{a} \times (\vec{b} \times \vec{c}) &= \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{vmatrix} \\
&= \left( \begin{array}{c}
a_2b_3 - a_3b_2 \\
a_3b_1 - a_1b_3 \\
-a_1b_2 + a_2b_1
\end{array} \right) \\
&= \langle a_2(b_1c_2 - b_2c_1) + a_3(b_1c_3 - b_3c_1), \\a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1), \\
-a_1(b_1c_3 - b_3c_1) - a_2(b_2c_3 - b_3c_2) \rangle
\end{align*}
\]

\[
= \langle a_2b_1c_2 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1, \\
a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1, \\
-a_1b_1c_3 + a_1b_3c_1 - a_2b_2c_3 + a_2b_3c_2 \rangle + \langle a_1b_1c_1, a_2b_2c_2, a_3b_3c_3 \rangle
\]

\[
= \langle (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1, \\
(a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2, \\
(a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 \rangle \\
+ \langle a_1b_1c_1, a_2b_2c_2, a_3b_3c_3 \rangle - \langle a_1b_1c_1, a_2b_2c_2, a_3b_3c_3 \rangle
\]

\[
= \langle (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1, \\
(a_1c_1 + a_2c_2 + a_3c_3)b_2 - (a_1b_1 + a_2b_2 + a_3b_3)c_2, \\
(a_1c_1 + a_2c_2 + a_3c_3)b_3 - (a_1b_1 + a_2b_2 + a_3b_3)c_3 \rangle
\]

\[
= \langle (\vec{a} \cdot \vec{c})b_1 - (\vec{a} \cdot \vec{b})c_1, \quad (\vec{a} \cdot \vec{c})b_2 - (\vec{a} \cdot \vec{b})c_2, \quad (\vec{a} \cdot \vec{c})b_3 - (\vec{a} \cdot \vec{b})c_3 \rangle
\]

\[
= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}
\]
The other expression in (1) follows from this one and the anti-commutativity of the cross product: 
\(-\vec{a} \times \vec{b} \times \vec{c} = \vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}\) so that 
\((\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}\).

For (2), we simply compute, using some basic facts about determinants:

\[
(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_3 = (\vec{c} \times \vec{a}) \cdot \vec{b} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}
\]

since each of the other two determinants is obtained from the first by 2 row interchanges, which are equal to \((-1)^2\) times the first.

The Jacobi identity (3) follows from the vector triple product: 
\((\vec{a} \times \vec{b}) \times \vec{c} + (\vec{c} \times \vec{a}) \times \vec{b} + (\vec{b} \times \vec{c}) \times \vec{a} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} + (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b} + (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a} = \vec{0}\).

\[\blacksquare\]
1.3 Geometric Properties of the Dot and Cross Products

The formulas

\[ \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta \] (1.35)

\[ |\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta \] (1.36)

for all vectors \( \vec{u} \) and \( \vec{v} \) in \( \mathbb{R}^3 \) forming an acute angle \( \theta \), already contain significant geometric information. The purely algebraic definition of \( \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 \), which a priori doesn’t say anything about angles and orthogonality, turns out to in fact give precisely that information. Indeed, we can use purely algebraic information about \( \vec{u} \) and \( \vec{v} \), namely their magnitudes and dot product, to gain the important geometric information about the acute angle they form:

\[ \theta = \cos^{-1} \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right) \] (1.37)

(provided, of course, that neither vector is the zero vector \( \vec{0} = (0,0,0) \), else we’d be dividing by 0).

This fact motivates the following definition. We say two vectors \( \vec{u} \) and \( \vec{v} \) are **orthogonal** or **perpendicular** if \( \vec{u} \cdot \vec{v} = 0 \), and we denote this by

\[ \vec{u} \perp \vec{v} \] (1.38)

From equation (1.35) we immediately get that

\[ \vec{u} \perp \vec{v} \iff \cos \theta = 0 \iff \theta = \frac{\pi}{2} \] (1.39)

Next, consider the projection of the vector \( \vec{u} \) onto the vector \( \vec{v} \), that is, drop a perpendicular from the arrowhead of \( \vec{u} \) onto the line containing \( \vec{v} \),
Observe that the length of the projection is clearly

\[
\text{comp}_\vec{v} \vec{u} = |\text{proj}_\vec{v} \vec{u}| = |\vec{u}| \cos \theta = \frac{|\vec{u}| |\vec{v}| \cos \theta = \vec{u} \cdot \vec{v}}{|\vec{v}|} \quad (1.40)
\]

Therefore, if we give it a direction, namely the unit direction \(\frac{\vec{v}}{|\vec{v}|}\) of \(\vec{v}\), we get

\[
\text{proj}_\vec{v} \vec{u} = \left( |\vec{u}| \cos \theta \right) \frac{\vec{v}}{|\vec{v}|^2} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \quad (1.41)
\]

Next, consider the parallelogram formed by the vectors \(\vec{u}\) and \(\vec{v}\):

We know that its area is the length of its base times its height,

\[
A = bh
\]

Now, \(b = |\vec{v}|\) and \(h = |\vec{u}| \sin \theta\), so by equation

\[
A = bh = |\vec{v}| |\vec{u}| \sin \theta = |\vec{u} \times \vec{v}| \quad (1.42)
\]

i.e. the area of the parallelogram determined by \(\vec{u}\) and \(\vec{v}\) is the length of their cross product!

Now consider a **parallelepiped** spanned by three vectors \(\vec{u}, \vec{v}, \vec{w}\), the 3-dimensional analog of the parallelogram, a rectangular solid whose opposite sides are all parallel.
The volume is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$ times the height $h$,

$$V = Ah = |\vec{u} \times \vec{v}| h$$

But note that $h$ is the length of the projection of $\vec{w}$ onto $\vec{u} \times \vec{v}$,

$$h = |\text{comp}_{\vec{u} \times \vec{v}} \vec{w}| = \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|} = \left| \vec{w} \cdot \left( \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} \right) \right| = |\vec{w}| |\cos \theta|$$

Hence,

$$V = Ah = |\vec{u} \times \vec{v}| \frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|} = |(\vec{u} \times \vec{v}) \cdot \vec{w}| \quad (1.43)$$

I.e., the volume $V$ is equal to the absolute value of the scalar triple product of $\vec{u}$, $\vec{v}$ and $\vec{w}$, which, by our formulas for the scalar triple product from Proposition 1.15, can be computed using the determinant: if $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then

$$V = |(\vec{u} \times \vec{v}) \cdot \vec{w}| = \text{absolute value of} \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \quad (1.44)$$
References

