The Axiom of Completeness for $\mathbb{R}$

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Contents

1 Axiom of Completeness

Definition 1 A collection $A \subseteq \mathbb{R}$ of real numbers is said to be **bounded above** by a real number M, the **upper bound**, if every number x in the collection A satisfies $x \leq M$. Let us write

$$\boxed{A^u \stackrel{\text{def}}{=} \{\text{all upper bounds of } A\}}$$

and let us write

$$\boxed{\sup A \stackrel{\text{def}}{=} \min A^u}$$

for the **supremum** or **least upper bound**, which is by definition the *minimum*, or *smallest* of the upper bounds, if it exists. ■

Axiom 1 *The* ***Axiom of Completeness (AoC)*** *postulates the existence of* $\min A^u$ *in* $\mathbb{R}$ *whenever* $A^u \neq \varnothing$ *and calls it the supremum,* $\sup A$. ■

Remark 1 We can form the analogous statement for greatest lower bound, but the existence of this turns out to be deducible from the existence of the least upper bound, so it doesn't need a separate axiom. ■

Definition 2 A collection $A \subseteq \mathbb{R}$ of real numbers is said to be **bounded below** by a real number M, the **lower bound**, if every number x in the collection A satisfies $M \leq x$. Let us write

$$\boxed{A^\ell \stackrel{\text{def}}{=} \{\text{all lower bounds of } A\}}$$

and let us write

$$\boxed{\inf A \stackrel{\text{def}}{=} \max A^\ell}$$

for the **infimum** or **greatest lower bound**, which is by definition the *maximum*, or *biggest* of the upper bounds, if it exists. ■

Proposition 2 (Existence of the Infimum) *AoC implies the existence of* $\inf A$ *for any nonempty set* $A \subseteq \mathbb{R}$ *satisfying* $A^\ell \neq \varnothing$. *In fact,*

$$\boxed{\inf A = -\sup(-A)}$$

where $-A \stackrel{\text{def}}{=} \{-a \in \mathbb{R} \mid a \in A\}$.

Proof: Since $A^\ell \neq \varnothing$, $\exists \ell \in A^\ell$ satisfying $\ell \leq a$ for all $a \in A$. Therefore $-a \leq -\ell$ for all $-a \in -A$, which shows that $(-A)^u \neq \varnothing$. By AoC, $s = \sup(-A) = \min(-A)^u$ exists in $\mathbb{R}$, so we just need to show that $-s = \inf A = \max A^\ell$: take any $\ell \in A^\ell$, then $-\ell \in (-A)^u$, so $s \leq -\ell$, which means $\ell \leq -s$, and indeed $-s = \max A^\ell = \inf A$. ■

Proposition 3 (Characterization of sup A) *Let $A \neq \varnothing$ and $A^u \neq \varnothing$. For any $s \in A^u$ we have*

$$\boxed{s = \sup A \iff \big(\forall \varepsilon > 0,\ \exists a \in A,\ s - \varepsilon < a\big)}$$

Proof: (i) $s = \sup A = \min A^u \implies \big(\forall \varepsilon > 0,\ s-\varepsilon \notin A^u\big) \implies (\exists a \in A,\ s-\varepsilon < a)$. (ii) If $s \in A^u$ and $\big(\forall \varepsilon > 0,\ \exists a \in A,\ s-\varepsilon < a\big)$, then, to see that $s = \min A^u$, consider any other $b \in \mathbb{R}$. We wish to prove that $b \in A^u \implies s \leq b$, by the contrapositive: If $s \not\leq b$, that is, if $s > b$, then $\varepsilon \stackrel{\text{def}}{=} s - b > 0$ satisfies $s - \varepsilon = b \notin A^u$ (because $\exists a \in A$, $s - \varepsilon < a$). ■

Proposition 4 (Characterization of inf A) *Let $A \neq \varnothing$ and $A^\ell \neq \varnothing$. For any $t \in A^\ell$ we have*

$$\boxed{t = \inf A \iff \big(\forall \varepsilon > 0,\ \exists a \in A,\ a < t + \varepsilon\big)}$$

Proof: (i) $t = \inf A = \max A^\ell \implies \big(\forall \varepsilon > 0,\ t+\varepsilon \notin A^\ell\big) \implies (\exists a \in A,\ a < t+\varepsilon)$. (ii) If $t \in A^\ell$ and $\big(\forall \varepsilon > 0,\ \exists a \in A,\ a < t+\varepsilon\big)$, then, to see that $t = \max A^\ell$, consider any other $b \in \mathbb{R}$. We wish to prove that $b \in A^\ell \implies b \leq t$, by the contrapositive: If $b \not\leq t$, that is, if $b > t$, then $\varepsilon \stackrel{\text{def}}{=} b - t > 0$ satisfies $t + \varepsilon = b \notin A^\ell$ (because $\exists a \in A$, $a < t + \varepsilon$). ■

Proposition 5 (Maximum vs Supremum) *Let $A \neq \varnothing$ and $A^u \neq \varnothing$. If $\max A$ exists, then*

$$\boxed{\max A = \sup A}$$

Proof: By definition, if $m = \max A$ exists, then $m \in A \cap A^u$. Therefore, $s = \sup A = \min A^u \leq m$, but since $m \in A$, we also have $m \leq s$, which together imply $m = s$ (by Axiom **2**.3 of $\mathbb{R}$). ■

Proposition 6 (Uniqueness of the Supemum and Infimum) *For any subset $A \subseteq \mathbb{R}$, the real numbers $\sup A$ and $\inf A$ are unique, if they exist.*

Proof: Suppose $s = \sup A = \min A^u$ exists, and that $t = \sup A = \min A^u$, too. Then $s = t$ because the minimum, when it exists, is unique: $s \leq t$ (because $s = \min A^u$ and $t \in A^u$) and $s \geq t$ (because $t = \min A^u$ and $s \in A^u$) together imply $s = t$ (by Axiom **2**.3 of $\mathbb{R}$), . Similarly, the maximum is unique: if $\ell, k = \inf A = \max A^\ell$, then $\ell \geq k$ and $\ell \leq k$ together imply $\ell = k$ (by Axiom **2**.3 of $\mathbb{R}$). ■

2 Monotone Convergence

Definition 3 A **convergent sequence** $(a_n)_{n\in\mathbb{N}}$ of real numbers, with **limit** $a \in \mathbb{R}$, satisfies

$$\boxed{\forall\varepsilon > 0,\ \exists N \in \mathbb{N},\ \big(n \geq N \implies |a_n - a| < \varepsilon\big)}$$ ■

Definition 4 A **bounded sequence** $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ satisfies

$$\boxed{\exists M > 0, \forall n \in \mathbb{N},\ |a_n| \leq M}$$ ■

Definition 5 A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is called **increasing** if

$$\boxed{a_n \leq a_{n+1}, \quad \forall n \in \mathbb{N}}$$

and called **decreasing** if

$$\boxed{a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N}}$$

If a sequence is either increasing or decreasing it is called **monotone**. ■

Axiom 2 *The* ***Monotone Convergence Theorem (MCT)****, as an axiom of* $\mathbb{R}$*, postulates the existence of a limit for every bounded monotone sequence in* $\mathbb{R}$*.* ■

Definition 6 A **subsequence** $(a_{n_k})_{k\in\mathbb{N}}$ of a real sequence $(a_n)_{n\in\mathbb{N}}$ is a new sequence derived from the original sequence by selecting a subset $\{a_{n_k} | k \in \mathbb{N}\}$ of its terms while maintaining their original relative order, that is by indexing them by an increasing sequence $n_1 < n_2 < n_3 < \cdots$. ■

Lemma 7 *Every sequence of real numbers has a monotonic subsequence.*

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, and call the m-th term **dominant**, or a **peak term**, if for all $n \geq m$ we have $a_n \leq a_m$. Either there are infinitely many dominant terms (example: $a_n = 1/n$) or there are finitely many (example: $a_1 = 1,\ a_n = 1 - 1/n$ for $n \geq 2$, has one dominant term). If there are infinitely many, let $(a_{n_k})_{k\in\mathbb{N}}$ be the subsequence consisting solely of dominant terms: it is clearly monotonic decreasing. If there are finitely many, let a_r denote the last dominant term and choose $n_1 > r$. Then, for all $N \geq n_1$, a_N is not dominant, so $\exists n_2 \geq N$ such that $a_{n_2} > a_{n_1}$. Repeat the process inductively: we have a monotonic increasing subsequence $(a_{n_k})_{k\in\mathbb{N}}$. ■

3 Nested Interval Property

Axiom 3 *The **Nested Interval Property (NIP)**, taken as an axiom of $\mathbb{R}$, postulates that every* nested *sequence of closed intervals of real numbers has* nonempty *intersection,*

$$\left.\begin{array}{ll}(1) & I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \\ (2) & I_n = [a_n,\, b_n] \subseteq \mathbb{R}\end{array}\right\} \implies \bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$$

■

4 Archimedean Property

Axiom 4 *The **Archimedean property (AP)** of $\mathbb{R}$ postulates, for any real number $x \in \mathbb{R}$, the existence of a larger natural number $n \in \mathbb{N}$,*

$$\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$$

■

5 Bolzano-Weierstrass Theorem

Axiom 5 *The **Bolzano-Weierstrass Theorem (BW)**, taken as an axiom of $\mathbb{R}$, postulates that every* bounded *sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers has a* convergent subsequence. ■

6 Cauchy Criterion

Definition 7 A **Cauchy sequence** $(a_n)_{n\in\mathbb{N}}$ of real numbers satisfies

$$\boxed{\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \big(n, m \geq N \implies |a_n - a_m| < \varepsilon\big)}$$

■

Axiom 6 *The **Cauchy Criterion (CC)** for $\mathbb{R}$ postulates the logical equivalence, for a real sequence, of being Cauchy and being convergent,*

$$(a_n)_{n\in\mathbb{N}} \text{ is convergent in } \mathbb{R} \iff (a_n)_{n\in\mathbb{N}} \text{ is Cauchy in } \mathbb{R}$$

■

Regardless of which of the axioms we take as our starting point, we have half of the CC equivalence as a mere consequence of our definitions of convergent sequences and Cauchy sequences:

Proposition 8 *Every convergent sequence is a Cauchy sequence.*

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence, with limit $\lim\limits_{n\to\infty} a_n = a$ in $\mathbb{R}$. Then, $\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ (n \geq N \implies |a_n - a| < \frac{\varepsilon}{2})$. If $n, m \geq N$, then

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a - a_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which shows that $(a_n)_{n\in\mathbb{N}}$ is Cauchy. ■

Proposition 9 *Every Cauchy sequence is bounded.*

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then, $\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ (m, n \geq N \implies |a_n - a_m| < \varepsilon)$. Let's take a concrete numerical value, say $\varepsilon = 1$, then

$$|a_n - a_N| < 1$$

for all $n \geq N$, for the appropriate N, which expands into the double inequality

$$\begin{aligned} -1 < a_n - a_N < 1 &\iff -|a_N| - 1 \leq a_N - 1 < a_n < a_N + 1 \leq |a_N| + 1 \\ &\iff |a_n| < |a_N| + 1 \end{aligned}$$

Thus, the tail of the sequence is bounded by $|a_N| + 1$. The first $N - 1$ terms, too, are bounded, by $M = \max\{|a_1|, \ldots, |a_{N-1}|\}$, so the whole sequence is bounded by $K \overset{\text{def}}{=} \max\{M,\ |a_N| + 1\}$. ■

Proposition 10 *If a Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} = a$, then the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent and $\lim\limits_{n\to\infty} a_n = a$.*

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence and suppose $\lim\limits_{k\to\infty} a_{n_k} = a$. Then,

- $\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \big(m, n \geq N \implies |a_n - a_m| < \dfrac{\varepsilon}{2}\big)$
- $\forall \varepsilon > 0,\ \exists M \in \mathbb{N},\ \big(k \geq M \implies |a_{n_k} - a| < \dfrac{\varepsilon}{2}\big)$

For any $\varepsilon > 0$, then, $\exists K \overset{\text{def}}{=} \max\{N, M\}$,

$$\begin{aligned} n, n_k \geq K \implies |a_n - a| &= |a_n - a_{n_k} + a_{n_k} - a| \\ &\leq |a_n - a_{n_k}| + |a_{n_k} - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

We conclude that $a_n \to a$. ■

7 Intermediate Value Theorem

Definition 8 Let $A \subseteq \mathbb{R}$. We call $B \subseteq A$ **(relatively) open in** A if $B = O \cap A$ for some $O \in \mathcal{T}_{\mathbb{R}}$, and we call $B \subseteq A$ **(relatively) closed in** A if $B = C \cap A$ for some $C \in \mathcal{C}_{\mathbb{R}}$. Let us denote the set of (relatively) open subsets of A by $\boxed{\mathcal{T}_A}$ and the set of (relatively) closed subsets of A by $\boxed{\mathcal{C}_A}$. ■

Let $f : A \to B$, where $A, B \subseteq \mathbb{R}$.

Definition 9 (Continuity) We say that f is **continuous at a point** $a \in A$, and write

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \big(|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon\,\big)}$$

We say f is **continuous on a set** A if it is continuous at all $a \in A$. Let

$$\boxed{C(A) \overset{\text{def}}{=} \{f : A \to \mathbb{R} \mid f \text{ is continuous on } A\}}$$

denote the **set of all continuous functions on** A. ■

Theorem 11 (Convergence Criterion for Continuous Functions) *Let $A \subseteq \mathbb{R}$, and let $f : A \to \mathbb{R}$. For any $a \in A$,*

$$\lim_{x \to a} f(x) = f(a) \iff \lim_{n \to \infty} f(a_n) = f(a) = f(\lim_{n \to \infty} a_n)$$

for all sequences $(a_n)_{n \in \mathbb{N}}$ in A with $a_n \to a$.

Proof: See Theorem 12, Lecture 10. ■

Axiom 7 *The **Intermediate Value Theorem (IVT)**, taken as an axiom of $\mathbb{R}$, postulates the existence of all intermediate y-values, between $f(a)$ and $f(b)$, for any continuous function f on a closed and bounded interval $[a, b]$,*

$$\exists c \in [a, b], \quad f(c) = y$$

whether $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$.

8 Equivalent Characterizations of the Completeness of $\mathbb{R}$

Theorem 12 (Completeness of $\mathbb{R}$) *The following are logically equivalent as axioms of* $\mathbb{R}$,

(1) *AoC* $(\Longrightarrow$ *AP*$)$

(2) *MCT* $(\Longrightarrow$ *AP*$)$

(3) *NIP + AP*

(4) *BW* $(\Longrightarrow$ *AP*$)$

(5) *CC + AP*

(6) *IVT*

Remark 13 I have tried to indicate above the fact that AoC, MCT and BW each implies AP, whereas NIP and CC do not imply AP, but require it as an ancillary assumption. See the counterexamples following the proof. ■

Proof:

(i) (1) $\Longrightarrow$ (2), or AoC $\Longrightarrow$ MCT (**Theorem 2.4.2** in Abbott): Suppose AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \quad \Longrightarrow s = \sup A = \min A^u$ exists) and let us deduce MCT (every bounded monotone sequence of real numbers converges).

If $(a_n)_{n\in\mathbb{N}}$ is an *increasing* bounded sequence in $\mathbb{R}$, then $A \stackrel{\text{def}}{=} \{a_n \mid n \in \mathbb{N}\}$ is nonempty and bounded, and we claim that

$$\lim_{n\to\infty} a_n = s \stackrel{\text{def}}{=} \sup A$$

By Proposition 3 we know $\big(\forall\varepsilon,\ \exists a_N \in A,\ s-\varepsilon < a_N\big)$, which means that

$$\begin{aligned}\exists N \in \mathbb{N},\ \big(n \geq N \quad &\Longrightarrow \quad a_N \leq a_n \quad (\textit{since } (a_n)_{n\in\mathbb{N}} \textit{ is increasing})\\ &\Longrightarrow \quad |a_n - s| = s - a_n \leq s - a_N < \varepsilon\big)\end{aligned}$$

and this shows that $\lim_{n\to\infty} a_n = s$ exists in $\mathbb{R}$.

If $(a_n)_{n\in\mathbb{N}}$ is *decreasing* and bounded, then $A \stackrel{\text{def}}{=} \{a_n \mid n \in \mathbb{N}\}$ is nonempty and bounded, and we claim that

$$\lim_{n\to\infty} a_n = t \stackrel{\text{def}}{=} \inf A$$

By Proposition 4 we know $\big(\forall\varepsilon,\ \exists a_N \in A,\ a_N < t+\varepsilon\big)$, which means that

$$\begin{aligned}\exists N \in \mathbb{N},\ \big(n \geq N \quad &\Longrightarrow \quad a_n \leq a_N \quad (\textit{since } (a_n)_{n\in\mathbb{N}} \textit{ is decreasing})\\ &\Longrightarrow \quad |a_n - t| = a_n - t \leq a_N - t < \varepsilon\big)\end{aligned}$$

and this shows that $\lim_{n\to\infty} a_n = t$ exists in $\mathbb{R}$. ■

(ii) (1) $\implies$ (3), or AoC $\implies$ NIP + AP (**Theorems 1.4.1-1.4.2** in Abbott): Suppose AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \implies s = \sup A = \min A^u$ exists) and let us deduce NIP (every nested sequence of closed intervals of real numbers has nonempty intersection) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$).

We first prove AoC implies NIP: Consider a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots$, where each $I_n = [a_n, b_n]$ is a closed subinterval of $\mathbb{R}$, and let us show that AoC implies $\bigcap_{n=1}^{n} I_n \neq \varnothing$. The nestedness of the intervals can be expressed as a sequence of inequalities:

$$a_1 \leq a_2 \leq \cdots \leq a_n \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$$

From this we conclude that the sets

$$A \stackrel{\text{def}}{=} \{a_n \mid n \in \mathbb{N}\}$$
$$B \stackrel{\text{def}}{=} \{b_n \mid n \in \mathbb{N}\}$$

are nonempty and bounded (by $M = \max\{|a_1|, |b_1|\}$), and moreover $B \subseteq A^u$ and $A \subseteq B^\ell$. Since we assumed AoC, we have assumed that

$$\sup A = \min A^u$$
$$\inf B = \max B^\ell$$

exist, and these must satisfy

$$a_n \leq \sup A \leq \inf B \leq b_n$$

for all $n \in \mathbb{N}$. (The middle inequality is shown as follows: $\sup A = \min A^u \leq b$ for all $b \in B$ since $B \subseteq A^u$, so $\sup A \in B^\ell$, and therefore $\sup A \leq \max B^\ell = \inf B$.) We conclude that $\exists \sup A, \inf B \in \bigcap_{n=1}^{\infty} I_n$, which is therefore nonempty. This proves NIP.

Next, we show that AoC implies AP: The key feature of $\mathbb{N}$ is that it possesses a successor function, according to the Peano axioms, $s(n) \stackrel{\text{def}}{=} n+1$ for all $n \in \mathbb{N}$. If $\mathbb{N}$ were bounded above, that is if $\mathbb{N}^u \neq \varnothing$, then AoC would guarantee the existence of $\alpha = \sup \mathbb{N} = \min \mathbb{N}^u$, which would consequently satisfy

$$\forall n \in \mathbb{N},\ n \leq \alpha \tag{8.1}$$

But then take $\varepsilon = 1$ and recall Proposition 3, which says that

$$\exists n \in \mathbb{N},\ \alpha - 1 < n$$

We would then be forced into a contradiction of (8.1):

$$\exists s(n) = n + 1 \in \mathbb{N},\ \alpha < n + 1 \tag{8.2}$$

Conclusion: $\mathbb{N}^u = \varnothing$, so $\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$.

For the second part of the AP, let $x > 0$, and use the first part of AP to find $n \in \mathbb{N}$ with $\frac{1}{x} < n$, which can be rewritten $\frac{1}{n} < x$. ■

(iii) (1) $\implies$ (4), or AoC $\implies$ BW, is **Exercise 2.5.9** in Abbott: Suppose AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \implies s = \sup A = \min A^u$ exists) and let us deduce BW (every bounded sequence has a convergent subsequence).

Suppose AoC and consider a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ bounded by $M > 0$. Then all terms a_n lie in the interval $[-M, M]$. Define

$$A \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid x < a_n \text{ for infinitely many } a_n\}$$

and note that $-M \in A$, so A is nonempty, while $M \in A^u$, so A^u is nonempty. AoC says $\exists s = \sup A = \min A^u$, and we claim that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ converging to s: By Proposition 3, $s = \sup A$ iff $(\forall \varepsilon > 0,\ \exists x \in A,\ s - \varepsilon < x)$, which here means

$$s - \varepsilon < x < \text{ infinitely many } a_n$$

For this same $\varepsilon > 0$, $s + \varepsilon \notin A$ (otherwise s would bound it, $s + \varepsilon \leq s$, an impossibility), meaning only finitely many $a_n > s + \varepsilon$. We conclude that

$$\text{infinitely many } a_n \text{ lie in } [s - \varepsilon, s + \varepsilon]$$

Let us now take $\varepsilon_k = \frac{1}{k}$ for each $k \in \mathbb{N}$, and let us form $I_k = [s - \frac{1}{k},\ s + \frac{1}{k}]$. Choose $a_{n_k} \in I_k$ from among the infinitely many terms lying in it, and observe that

$$|a_{n_k} - s| \leq \frac{2}{k}$$

which we could make less than any $\varepsilon > 0$ by using AP to find $k > \frac{2}{\varepsilon}$. This shows that $a_{n_k} \to s$. ■

(iv) (1) $\implies$ ((4) $\implies$) (5), or AoC $\implies$ (BW $\implies$) CC: Suppose AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \implies s = \sup A = \min A^u$ exists) and let us deduce CC (a sequence of real numbers converges iff it is Cauchy).

We already saw (Proposition 8) that, directly from the definitions, a convergent sequence must be Cauchy. We must therefore prove that, if we assume AoC, then every Cauchy sequence in $\mathbb{R}$ must converge. There are several routes toward this result, we indicate one. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. Then by Proposition 9 it is bounded, say by $M > 0$. We can now repeat verbatim the preceding proof that AoC (together with its consequent AP) implies that bounded sequences must have a convergent subsequence, or we can merely cite BW (which AoC implies) and say that $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(a_{n_k})_{k\in\mathbb{N}}$, say with limit a in $\mathbb{R}$. This recalls Proposition 10, which says that $(a_n)_{n\in\mathbb{N}}$ itself converges to a. ■

(v) (2) $\implies$ AP and (1), or MCT $\implies$ AP + AoC, is **Exercise 2.4.4** in Abbott: Suppose MCT (every bounded monotone sequence of real numbers converges), and let us deduce AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \implies s = \sup A = \min A^u$ exists).

We first prove MCT implies AP: Let $x \in \mathbb{R}$. If $x \leq 0$, then any $n \in \mathbb{N}$ with $n \geq 1$ will do, so suppose $x > 0$. Now, if $x \in \mathbb{N}^u$, then $\forall n \in \mathbb{N},\ n \leq x$, so we can consider the sequence $a_n = n$, which is increasing and bounded. Our assumption, the MCT, then says that this sequence converges, say to $L \in \mathbb{R}$, meaning

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \big(n \geq N \implies |n - L| < \varepsilon\big)$$

Since this is true for all ε, let us take a specific numerical value, say $\varepsilon = 1$. The inequality $|n - L| < 1$ expands into the double inequality

$$-1 < n - L < 1 \iff \overbrace{L - 1 < n}^{\text{note}} < L + 1$$

This is true for all $n \geq N$ for the appropriate $N \in \mathbb{N}$, so it is true in particular for $n + 2 > n \geq N$:

$$L - 1 < n + 2 < L + 1 \implies L - 3 < \underbrace{n < L - 1}_{\text{note}}$$

We have reached the contradiction $L - 1 < n < L - 1$, so we conclude that $x \notin \mathbb{N}^u$, and instead that $\exists n \in \mathbb{N},\ n > x$.

We next prove MCT+AP implies AoC: Assume MCT and let $\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing$. We will demonstrate that $\sup A = \min A^u$ exists in $\mathbb{R}$. Choose

$$a_1 \in A$$
$$b_1 \in A^u$$

If $a_1 = b_1$, then $s \stackrel{\text{def}}{=} a_1 = \max A$, which therefore must equal $\min A^u = \sup A$ by Proposition 5, and we are done. Otherwise, if $a_1 < b_1$, take the midpoint between them,

$$c_1 \stackrel{\text{def}}{=} \frac{a_1 + b_1}{2}$$

and consider the two cases:

(a) **Case 1:** $[c_1 \cap b_1) \cap A \neq \varnothing$: In this case choose $a_2 \in [c_1, b_1) \cap A$ and let $b_2 = b_1$.

(b) **Case 2:** $[c_1 \cap b_1) \cap A = \varnothing$: In this case, we let $a_2 = a_1$ and choose $b_2 \in [c_1, b_1) \subseteq A^u$.

Either way, we have

$$a_1 \leq a_2 \in A$$
$$b_2 \leq b_1 \in A^u$$

Use the above procedure to define an algorithm which produces $a_{n+1} \in A$ and $b_{n+1} \in A^u$ from $a_n \in A$ and $b_n \in A^u$, and notice that $(a_n)_{n\in\mathbb{N}}$ is increasing and $(b_n)_{n\in\mathbb{N}}$ is decreasing, and both are bounded (by $\max\{|a_1|, |b_1|\}$). MCT says the sequences converge in $\mathbb{R}$:

$$\lim_{n\to\infty} a_n = a$$
$$\lim_{n\to\infty} b_n = b$$

Moreover, since $b_n - c_n = c_n - a_n = \frac{1}{2}(b_n - a_n)$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} b_{n+1} - a_{n+1} &\leq \frac{1}{2}(b_n - a_n) \\ &\leq \frac{1}{2^2}(b_{n-1} - a_{n-1}) \\ &\vdots \\ &\leq \frac{1}{2^n}(b_1 - a_1) \end{aligned}$$

Since $\frac{1}{2^n} < \frac{1}{n}$ for all $n \in \mathbb{N}$, for every ε we can use the AP to find an $N \in \mathbb{N}$ for which

$$n \geq N \implies \frac{1}{n} \leq \frac{1}{N} < \frac{\varepsilon}{3(b_1 - a_1)}$$

We conclude that

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \big(n \geq N \implies |b_{n+1} - a_{n+1}| < \frac{\varepsilon}{3}\big)$$

If, for this same ε, we find $M \in \mathbb{N}$ so large that for $n \geq \max\{N, M\}$ we have $|b - b_{n+1}| < \frac{\varepsilon}{3}$ and $|b - b_{n+1}| < \frac{\varepsilon}{3}$, we finally reach:

$$\begin{aligned} |b - a| &= |b - b_{n+1} + b_{n+1} - a_{n+1} + a_{n+1} - a| \\ &\leq |b - b_{n+1}| + |b_{n+1} - a_{n+1}| + |a_{n+1} - a| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Since this is true for all $\varepsilon > 0$, we have that $a = b$. We claim that this common value is the supremum of A,

$$s \stackrel{\text{def}}{=} \sup A = a = b$$

To see this, use Proposition 3: Let $\varepsilon > 0$, and use the above results to get that $\exists N \in \mathbb{N}$, $(n \geq N \implies a - a_n < \varepsilon)$, which implies that $a - \varepsilon < a_n$. Since $a_n \in A$, this shows that $a = b = s = \sup A$. ■

Remark 14 Concerning our algorithm, we should include a line in the inductive step: if $a_n = b_n$ for some $n \in \mathbb{N}$ in the procedure, we should stop the algorithm, because $a_n = b_n \in A \cap A^u \implies a_n = b_n = \max A = \min A^u = \sup A$. ■

(vi) (2) $\implies$ (3), or MCT $\implies$ NIP+ AP, is **Exercise 2.4.4** in Abbott: Suppose MCT (every bounded monotone sequence of real numbers converges), and let us deduce NIP (every nested sequence of closed intervals of real numbers has nonempty intersection) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$).

We first prove MCT implies NIP: Consider a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots$, where each $I_n = [a_n, b_n]$ is a closed subinterval of $\mathbb{R}$, and let us show that MCT implies $\bigcap_{n=1}^{n} I_n \neq \varnothing$. The nestedness of the intervals can be expressed as a sequence of inequalities:

$$a_1 \leq a_2 \leq \cdots \leq a_n \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$$

From this we conclude that $(a_n)_{n\in\mathbb{N}}$ is increasing, $(b_n)_{n\in\mathbb{N}}$ is decreasing, and both are bounded (by $M = \max\{|a_1|, |b_1|\}$). MCT applies to give us the existence of the limits

$$\lim_{n\to\infty} a_n = a$$
$$\lim_{n\to\infty} a_n = b$$

in $\mathbb{R}$. The Order Limit Laws say $a \leq b$, and this in combination with the facts $a_n \leq a$ and $b_n \geq b$ give

$$a_n \leq a \leq b \leq b_n$$

for all $n \in \mathbb{N}$. Therefore, $\exists a, b \in \bigcap_{n=1}^{\infty} I_n$, and the intersection is nonempty.

We next prove MCT implies AP: Let $x \in \mathbb{R}$, and suppose by way of contradiction that $x \in \mathbb{N}^u$, i.e. $n \leq x$ for all $n \in \mathbb{N}$. Then the increasing sequence $a_n = n$ is bounded by x, so MCT applies to give the existence of a limit, $\lim_{n\to\infty} n = L$:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ (n \geq N \implies |n - L| < \varepsilon)$$

Expanded to a double inequality, this says $L - \varepsilon < n < L + \varepsilon$ for all $n \geq N$ and all $\varepsilon > 0$. Taking $\varepsilon = 1$, this says all $n \geq N$ lie in the interval $(L-1,\ L+1)$ of length 2, an impossibility (since, e.g., n and $n+3$ are a distance of 3 apart yet lie in an interval of length 2). ■

(vii) (2) $\implies$ (4), or MCT $\implies$ BW, is **Exercise 2.5.8** in Abbott: Suppose MCT (every bounded monotone sequence of real numbers converges), and let us deduce BW (every bounded sequence has a convergent subsequence).

> By Lemma 7 we know that every real sequence $(a_n)_{n\in\mathbb{N}}$ (bounded or undbounded) has a monotonic subsequence $(a_{n_k})_{k\in\mathbb{N}}$. If $(a_n)_{n\in\mathbb{N}}$ is bounded, then so is $(a_{n_k})_{k\in\mathbb{N}}$, and the MCT assumption applies: $(a_{n_k})_{k\in\mathbb{N}}$ converges to a limit $a \in \mathbb{R}$. ■

(viii) (2) $\implies$ (5), or MCT $\implies$ CC + AP: Suppose MCT (every bounded monotone sequence of real numbers converges), and let us deduce CC (a sequence of real numbers converges iff it is Cauchy). [We have already shown, two proofs above, that MCT implies AP.]

> Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}$. By Proposition 9 we know it is bounded, while by Lemma 7 we know it possesses a monotone subsequence $(a_{n_k})_{k\in\mathbb{N}}$. Our MCT assumption says $(a_{n_k})_{k\in\mathbb{N}}$ converges, say to a in $\mathbb{R}$. By Proposition 10 the original sequence itself must also converge to a. ■

(ix) (3) $\implies$ (1), or NIP + AP $\implies$ AoC, is **Exercise 2.5.4** in Abbott: Suppose NIP (every nested sequence of closed intervals of real numbers has nonempty intersection) and AP ($\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $x < n$, and $\forall x > 0$, $\exists n \in \mathbb{N}$, $0 < \frac{1}{n} < x$) and let us deduce AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \implies s = \sup A = \min A^u$ exists).

> Let $\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing$, and let us show that $s \stackrel{\text{def}}{=} \sup A$ exists in $\mathbb{R}$. Since we are assuming NIP, we describe an algorithm which constructs a nested sequence of closed intervals from A and A^u as follows: choose $a_1 \in A$ and $b_1 \in A^u$, which automatically satisfy $a_1 \leq b_1$, and let $I_1 \stackrel{\text{def}}{=} [a_1, b_1]$. If $a_1 = b_1 \in A \cap A^u$, then $s = a_1 = b_1 = \max A = \min A^u = \sup A$ exists in $\mathbb{R}$ by Proposition 5, and we are done. Otherwise, if $a_1 < b_1$, take the midpoint $c_1 \stackrel{\text{def}}{=} \dfrac{a_1 + b_1}{2}$ of I_1 and see whether $c_1 \in A$ or in A^u–if $c_1 \in A \cap A^u$, then again $c = \max A = \min A^u = \sup A$ exists by Proposition 5 and we are done. Otherwise, if $c_1 \in A^u - A$, then let $a_2 = a_1$ and $b_2 = c_1$. If $c_1 \in A - A^u$, then let $a_2 = c_1$ and $b_2 = b_1$. Either way, we form $I_2 \stackrel{\text{def}}{=} [a_2, b_2]$. Repeat inductively, stopping the algorithm if at any step $a_n = b_n$, since then we have found our supremum of A. If the algorithm runs to infinity, then we have constructed a countable collection of nested closed intervals
>
> $$I_1 \supseteq I_2 \supseteq \cdots$$
>
> The intersection $\bigcap_{n=1}^{\infty} I_n$ is therefore nonempty, by NIP.

To see that this intersection consists of only one point, s, we invoke AP: Since

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) = \cdots = \frac{1}{2^n}(b_1 - a_1)$$

and since $\frac{1}{2^n} < \frac{1}{n}$ for all $n \in \mathbb{N}$, we have that $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $(n \geq N \implies \frac{1}{n} < \frac{\varepsilon}{b_1 - a_1}$. This shows that

$$\lim_{n\to\infty} |b_{n+1} - a_{n+1}| = 0$$

and this in turn shows that the intersection consists of only one point, $\bigcap_{n=1}^{\infty} I_n = \{s\}$ (if there were at least two points $a, b \in \bigcap_{n=1}^{\infty} I_n$, say $a < b$, then we could take $\varepsilon = \frac{b-a}{2}$ and find an interval I_n of smaller length than the distance from a to b, an impossibility).

Finally, to see that $s = \sup A$, note that $s \in \bigcap_{n=1}^{\infty} I_n$ means $a_n \leq s \leq b_n$ for all $n \in \mathbb{N}$, which means $s \in A^u$. But for any $b \in A^u$ we also have $s \leq b$, for if $b < s$, then we could take $\varepsilon = \frac{s-b}{2}$ and reach the following contradiction: for this ε, there is an $n \in \mathbb{N}$ such that

$$|a_n - s| \leq |a_n - b_n| < \varepsilon = \frac{s-b}{2}$$

or $\frac{b-s}{2} < a_n - s < \frac{s-b}{2}$. The first inequality implies $b < \frac{b+s}{2} < a_n$, contradicting the fact that $b \in A^u$. ■

(x) (3) $\implies$ (2), or NIP + AP $\implies$ MCT: Suppose NIP (every nested sequence of closed intervals of real numbers has nonempty intersection) and AP ($\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$, $x < n$, and $\forall x > 0$, $\exists n \in \mathbb{N}$, $0 < \frac{1}{n} < x$) and let us deduce MCT (every bounded monotone sequence of real numbers converges).

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded monotone sequence of real numbers. A direct proof of its convergence would be a repetition of NIP + AP $\implies$ AoC $\implies$ MCT proofs, for we would let $A \stackrel{\text{def}}{=} \{a_n \in \mathbb{R} \mid n \in \mathbb{N}\}$, which would satisfy $A \neq \varnothing$, $A^u \neq \varnothing$ and $A^\ell \neq \varnothing$, since $a_n \in A$, $M \in A^u$, $-M \in A^\ell$.

If a_n is increasing, then by letting $a_1 = a_1$, $b_1 = M$ and $c_1 = \frac{a_1+b_1}{2}$ and checking whether $c_1 \in A$ or A^u, then letting $a_2 = c_1$ in the first case, $b_2 = c_1$ in the second case, and repeating, we would get nested intervals $I_n = [a_n, b_n]$ whose intersection would be nonempty and in fact, by AP, consisting of a single point s, namely $s = \sup A$. (See the last proof above.) Once we established the existence of $s = \sup A$, we would prove that $\lim_{n\to\infty} a_n = s$ as in the proof of AoC $\implies$ MCT, by using Proposition 3. If a_n is decreasing, we would similarly establish the existence of $t = \inf A$ and apply Proposition 4 to show $\lim_{n\to\infty} a_n = t$.

In short, we can just cite the fact that NIP + AP imply AoC, which in turn implies MCT, but these two implications can be made explicit upon demand. ■

(xi) (3) $\implies$ (4), or NIP + AP $\implies$ BW: Suppose NIP (every nested sequence of closed intervals of real numbers has nonempty intersection) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and let us deduce BW (every bounded sequence of real numbers has a convergent subsequence).

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence with $|a_n| \leq M$ for all $n \in \mathbb{N}$, and let us show that some subsequence $(a_{n_k})_{k\in\mathbb{N}}$ converges. The inequality $|a_n| \leq M$ expands into the double inequality

$$-M \leq a_n \leq M$$

for all $n \in \mathbb{N}$. Let $a_0 = -M$, $b_0 = M$, and $I_0 = [a_0, b_0]$. Divide I_0 into two halves $[-M, 0]$ and $[0, M]$ and choose for $I_1 = [a_1, b_1]$ that which contains infinitely many terms a_n of the sequence (if both contain infinitely many terms, choose either). Then choose $a_{n_1} \in I_1$ from among the infinite choices. Next, bisect I_1 and choose for I_2 the half which contains infinitely many terms a_n, and pick $a_{n_2} \in I_2$ from among the infinitely many choices. Repeat inductively. Since

$$I_1 \supseteq I_2 \supseteq \cdots \implies a_{n_1} \leq a_{n_2} \leq \cdots$$

we have both a sequence of nested closed intervals and an increasing subsequence $(a_{n_k})_{k\in\mathbb{N}}$. By our NIP assumption, $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$, while by our AP assumption, $|b_n - a_n| < \frac{M}{2^{n-1}} < \frac{1}{n-1}$ can be made less than any $\varepsilon > 0$ by finding $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon} + 1$ and taking $n \geq N$. Together, these show that $\bigcap_{n=1}^{\infty} I_n$ consists of a single point a, the limit of the subsequence, since $|a_n - a| \leq |b_n - a_n| < \varepsilon$ for large enough n. ■

(xii) (3) $\implies$ (5), or NIP + AP $\implies$ CC: Suppose NIP (every nested sequence of closed intervals of real numbers has nonempty intersection) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and let us deduce CC (a sequence of real numbers converges iff it is Cauchy).

That a convergent sequence is Cauchy was proved in Proposition 8, so suppose $(a_n)_{n\in\mathbb{N}}$ is Cauchy and let us show that NIP + AP imply its convergence. First, Proposition 9 implies the sequence is bounded: $\forall n \in \mathbb{N},\ \exists M > 0, (n \geq N \implies |a_n| \leq M)$. Let $I_1 = [-M, M]$. Then, by taking $\varepsilon_k = \frac{1}{k}$, the Cauchy property says that $\exists N_k \in \mathbb{N},\ (n \geq N \implies |a_n - a_{N_k}| < \frac{1}{k})$, which means that

$$n \geq N_k \implies a_{N_k} - \frac{1}{k} < a_n < a_{N_k} + \frac{1}{k}$$

Define $I_k = [a_{N_k} - \frac{1}{k},\ a_{N_k} + \frac{1}{k}] \cap I_{k-1}$ recursively, and obtain a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \cdots$. The NIP says $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$, while AP says the intersection consists of a single point a (because if $I_k = [\alpha_k, \beta_k]$, then $|\alpha_k - \beta_k| < \frac{1}{k}$, which shows the lengths of the intervals I_k tend to 0). Since $|a_n - a| \leq |\alpha_k - \beta_k| < \frac{1}{k}$, we have $a_n \to a$. ■

(xiii) (4) $\implies$ (1), or BW $\implies$ AP + AoC: Suppose BW (every bounded sequence of real numbers has a convergent subsequence) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and let us deduce AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \implies s = \sup A = \min A^u$ exists).

Let $\varnothing \neq A \subseteq \mathbb{R}$ and suppose $A^u \neq \varnothing$. We use the same algorithm as in the proof of (5) $\implies$ (1) below, which constructs Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ in A and $(b_n)_{n\in\mathbb{N}}$ in A^u, satisfying

$$a_1 \leq \cdots a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq \cdots \leq b_1$$

for all $n \in \mathbb{N}$:

(1) Choose $a_1 \in A \neq \varnothing$ and $b_1 \in A^u \neq \varnothing$, which automatically satisfy $a_1 \leq b_1$.
   (a) If $a_1 = b_1 \in A \cap A^u$, then $s = a_1 = b_1 = \max A = \min A^u = \sup A$ exists in $\mathbb{R}$ by Proposition 5, and we are done.
   (b) If $a_1 < b_1$, take the midpoint $c_1 \overset{\text{def}}{=} \frac{a_1+b_1}{2}$ between a_1 and b_1 and see whether $c_1 \in A$ or in A^u.

(2) (a) If $c_1 \in A \cap A^u$, then again $c = \max A = \min A^u = \sup A$ exists by Proposition 5 and we are done.
   (b) If $c_1 \in A^u - A$, then let $a_2 = a_1$ and $b_2 = c_1$. If $c_1 \in A - A^u$, then let $a_2 = c_1$ and $b_2 = b_1$.

(3) Repeat inductively, stopping the algorithm if at any step $a_n = b_n$, since then we have found our supremum of A.

If the algorithm runs to infinity, then we have constructed two Cauchy sequences, $(a_n)_{n\in\mathbb{N}}$ in A and $(b_n)_{n\in\mathbb{N}}$ in A^u, the first increasing, the second decreasing: let $\varepsilon > 0$ and use AP to find $N \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N \quad &\implies \quad \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \frac{\varepsilon}{4(b_1 - a_1)} \\ &\implies \quad |a_{n+1} - b_{n+1}| < \frac{b_1 - a_1}{2^n} < \frac{\varepsilon}{4} \end{aligned}$$

Therefore, for all $m = n + k \geq n \geq N$, we have

$$\begin{aligned} |a_m - a_n| &= |a_{n+k} - b_{n+k} + b_{n+k} - a_n| \\ &\leq |a_{n+k} - b_{n+k}| + |b_{n+k} - a_n| \\ &\leq |a_{n+k} - b_{n+k}| + |b_n - a_n| \\ &< \frac{b_1 - a_1}{2^{n+k}} + \frac{b_1 - a_1}{2^{n-1}} \\ &\leq \frac{b_1 - a_1}{2^{n-2}} \\ &< \varepsilon \end{aligned}$$

Similarly, $|b_m - b_n| < \frac{b_1 - a_1}{2^{n-2}} < \varepsilon$ for all $m = n + k \geq n \geq N$. The sequences are Cauchy.

Cauchy sequences are bounded (Proposition 9), so our BW assumption says they have convergent subsequences, $a_{n_k} \to a$ and $b_{n_k} \to b$.

By Proposition 10, if a subsequence of a Cauchy sequences converges to a (or b in the second case), then the original sequence converges to a, too, $a_n \to a$ (and $b_n \to b$).

Lastly, by the order limit laws, we know $a \leq b$. We claim that in fact $a = b = \sup A$. By a slight modification of the application of AP, we can show that $|a_n - b_n| < \frac{b_1 - a_1}{2^n} < \varepsilon$, which shows $\lim_{n\to\infty} |a_n - b_n| = 0$. Since $a_n \leq a \leq b \leq b_n$, the limit laws say

$$0 \leq |a - b| \leq \lim_{n\to\infty} |a_n - b_n| = 0$$

which means $|a - b| = 0$, and therefore $a = b$. To see that $a = b = \sup A$, use Proposition 3: Let $\varepsilon > 0$, and use the definition of convergence to get that $\exists N \in \mathbb{N},\ (n \geq N \implies a - a_n < \varepsilon)$, which implies that $a - \varepsilon < a_n$. Since $a_n \in A$, this shows that $a = b = s = \sup A$. ■

(xiv) (4) $\implies$ (2), or BW $\implies$ MCT, is **Exercise 2.6.7(a)** in Abbott: Suppose BW (every bounded sequence of real numbers has a convergent subsequence) and let us deduce AP and MCT (every bounded monotone sequence of real numbers converges).

We first show that BW implies AP, which we will need to show BW implies MCT: If $x \leq 0$, then any $n \in \mathbb{N}$ will do to show $n > x$, so suppose $x > 0$. Suppose AP fails to hold, suppose $\exists x \in \mathbb{N}^u$, then let $a_n \stackrel{\text{def}}{=} n$ and note that $|a_n| \leq x$ for all $n \in \mathbb{N}$, so our sequence is bounded. BW says a subsequence $a_{n_k} = n_k$ (which is still increasing) converges, say to a. Thus,

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ |a_{n_k} - a| < \varepsilon$$

Taking a concrete numerical value, say $\varepsilon = 1$, we conclude that

$$-1 < n_k - a < 1 \implies a - 1 < n_k < a + 1$$

for all $n_k \in \mathbb{N}$. Now, $n_{k+2} \geq n_k + 2$ yet also satisfies this:

$$-1 < n_{k+2} - a < 1 \implies a - 1 < \overbrace{n_{k+2} < a + 1}^{\text{note}}$$

and also

$$\underbrace{a + 1 = 2 + (a - 1) < 2 + n_k \leq n_{k+2}}_{\text{note}}$$

which is a contradiction. We conclude that $x \notin \mathbb{N}^u$ but instead $\mathbb{N}^u = \varnothing$, and therefore $\exists n \in \mathbb{N},\ n > x$.

We now show that BW implies MCT: Let $(a_n)_{n\in\mathbb{N}}$ be a bounded monotonic sequence of real numbers. First of all, this sequence is Cauchy: We assumed $\boxed{a_n \leq M}$ for all $n \in \mathbb{N}$, but if $(a_n)_{n\in\mathbb{N}}$ is not Cauchy, then

$$\exists \varepsilon > 0, \forall N \in \mathbb{N}, (\exists n, m \geq N,\ |a_n - a_m| \geq \varepsilon)$$

Let $N_1 \stackrel{\text{def}}{=} N$. Since a_n is increasing, we may suppose WOLOG $n_1 > m_1 \geq N_1$, so that $a_{n_1} \geq a_{m_1}$ and therefore

$$a_{n_1} - a_{m_1} \geq \varepsilon$$

Next, choose $N_2 > n_1$, and find $n_2 > m_2 \geq N_2$ satisfying

$$a_{n_2} - a_{m_2} \geq \varepsilon$$

Repeat inductively, and observe that we obtain a sequence of natural numbers

$$m_1 < n_1 < m_2 < n_2 \cdots$$

whose corresponding sequential terms form intervals $I_k = (a_{m_k}, a_{n_k})$ of length $\geq \varepsilon$, $\forall k \in \mathbb{N}$. But by AP, $\exists k \in \mathbb{N}$ so large that

$$k > \frac{M - a_{m_1}}{\varepsilon}$$

and for this k we have

$$\begin{aligned} a_{n_k} - a_{m_1} &\geq k \text{ lengths of the intervals } I_k \\ &\geq k\varepsilon \\ &> \frac{M - a_{m_1}}{\varepsilon} \cdot \varepsilon \\ &= M - a_{m_1} \end{aligned}$$

which shows $\boxed{a_{n_k} > M}$, a contradiction. We conclude that $(a_n)_{n\in\mathbb{N}}$ is Cauchy. The decreasing case is handled similarly, but using $-M$ instead, and finding $-M \leq a_{n_k} < M$.

Secondly, since $(a_n)_{n\in\mathbb{N}}$ is assumed bounded, our BW assumption guarantees the existence of a convergent subsequence, $a_{n_k} \to a$. Proposition 10 then tells us that $a_n \to a$, too, which shows that BW implies MCT. ■

(xv) (4) $\implies$ (3), or BW $\implies$ NIP + AP: Suppose BW (every bounded sequence of real numbers has a convergent subsequence) and let us deduce NIP (every nested sequence of closed intervals of real numbers has nonempty intersection) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$).

> Consider a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \cdots$, where $I_n = [a_n, b_n]$, and let us use CC + AP to show that $\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$. Then $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$, so $(a_n)_{n\in\mathbb{N}}$ is increasing and bounded, $(b_n)_{n\in\mathbb{N}}$ is decreasing and bounded. Exactly as in the proof of (4) $\implies$ (2), we conclude that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy, and therefore converge, $a_n \to a$ and $b_n \to b$, with $a \leq b$ by the order limit laws. Since $a_n \leq a \leq b \leq b_n$ for all n, we conclude that $a, b \in \bigcap_{n\in\mathbb{N}}$—possibly $a = b$—which is therefore nonempty. That BW implies AP was proved in the course of showing (4) $\implies$ (2). ■

(xvi) (4) $\implies$ (5), or BW $\implies$ CC: Suppose BW (every bounded sequence of real numbers has a convergent subsequence) and let us deduce CC (a sequence of real numbers converges iff it is Cauchy).

> Any convergent sequence is Cauchy (Proposition 8), so we need only show that a Cauchy sequence converges. Suppose $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. By Proposition 9 it is bounded, so by our BW assumption it has a convergent subsequence, $a_{n_k} \to a$. Proposition 10 then guarantees that $a_n \to a$, too. ■

(xvii) (5) $\implies$ (1), or CC + AP $\implies$ AoC, is **Theorem 2.6.4** in Abbott: Suppose CC (a sequence of real numbers converges iff it is Cauchy) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and let us deduce AoC ($\varnothing \neq A \subseteq \mathbb{R}$ with $A^u \neq \varnothing \implies s = \sup A = \min A^u$ exists).

> Let $\varnothing \neq A \subseteq \mathbb{R}$ and let $A^u \neq \varnothing$. We use the same algorithm as in the proof of (4) $\implies$ (1) above, which constructs Cauchy sequences $(a_n)_{n\in\mathbb{N}}$ in A and $(b_n)_{n\in\mathbb{N}}$ in A^u, satisfying
>
> $$a_1 \leq \cdots a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq \cdots \leq b_1$$
>
> for all $n \in \mathbb{N}$. We will not repeat the details here. By our CC assumption, these Cauchy sequences must converge, say $a_n \to a$, $b_n \to b$. By the order limit laws, $a \leq b$. We claim that in fact $a = b = \sup A$. An application of AP as in the proof of (4) $\implies$ (1) shows that $|a_n - b_n| < \frac{b_1 - a_1}{2^n} < \varepsilon$, which shows $\lim_{n\to\infty} |a_n - b_n| = 0$. Since $a_n \leq a \leq b \leq b_n$, the limit laws say $0 \leq |a - b| \leq \lim_{n\to\infty} |a_n - b_n| = 0$, which means $|a - b| = 0$, and therefore $a = b$. To see that $a = b = \sup A$, use Proposition 3: Let $\varepsilon > 0$, and use the definition of convergence to get that $\exists N \in \mathbb{N},\ (n \geq N \implies a - a_n < \varepsilon)$, which implies that $a - \varepsilon < a_n$. Since $a_n \in A$, this shows that $a = b = s = \sup A$. ■

(xviii) (5) $\implies$ (2), or CC + AP $\implies$ MCT: Suppose CC (a sequence of real numbers converges iff it is Cauchy) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and let us deduce MCT (every bounded monotone sequence of real numbers converges).

> Let $(a_n)_{n\in\mathbb{N}}$ be a bounded (say by M) monotone sequence in $\mathbb{R}$. The same proof as in (4) $\implies$ (2) shows that $(a_n)_{n\in\mathbb{N}}$ is Cauchy. Our CC assumption will imply that it converges, say to $a \in \mathbb{R}$. ■

(xix) (5) $\implies$ (3), or CC + AP $\implies$ NIP + AP: Suppose CC (a sequence of real numbers converges iff it is Cauchy) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and let us deduce NIP (every nested sequence of closed intervals of real numbers has nonempty intersection), naturally + AP.

> Consider a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \cdots$, where $I_n = [a_n, b_n]$, and let us use CC + AP to show that $\bigcap_{n\in\mathbb{N}} I_n \neq \varnothing$. Then $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$, so $(a_n)_{n\in\mathbb{N}}$ is increasing and bounded, $(b_n)_{n\in\mathbb{N}}$ is decreasing and bounded. As in the proof of (5) $\implies$ (2), or CC + AP implies MCT, we conclude that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy, and therefore converge, $a_n \to a$ and $b_n \to b$, with $a \leq b$ by the order limit laws. Since $a_n \leq a \leq b \leq b_n$ for all n, we conclude that $a, b \in \bigcap_{n\in\mathbb{N}}$—possibly $a = b$—which is therefore nonempty. ■

(xx) (5) $\implies$ (4), or CC + AP $\implies$ BW, is **Exercise 2.6.7(b)** in Abbott: Suppose CC (a sequence of real numbers converges iff it is Cauchy) and AP ($\forall x \in \mathbb{R},\ \exists n \in \mathbb{N},\ x < n$, and $\forall x > 0,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < x$) and let us deduce BW (every bounded sequence of real numbers has a convergent subsequence).

> Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. By Lemma 7, $(a_n)_{n\in\mathbb{N}}$ has a (necessarily bounded) monotonic subsequence $(a_{n_k})_{k\in\mathbb{N}}$, whether increasing or decreasing. We could now repeat the proof of (5) $\implies$ (2), or CC + AP implies MCT to show that this subsequence is Cauchy, and hence converges by our CC + AP assumptions, or we can merely cite the fact that CC + AP implies MCT which implies that $a_{n_k} \to a$. ■