11: Continuous Functions on $\mathbb{R}$
Part II

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Contents

1 Continuous Functions on Compact Subsets of $\mathbb{R}$

1.1 Preservation of Compactness and Extreme Value Theorem (EVT)

Theorem 1 (Continuous Functions Preserve Compactness) *If $K \in \mathcal{K}_{\mathbb{R}}$ and $f \in C(\mathbb{R})$, then $f(K) \in \mathcal{K}_{\mathbb{R}}$.*

Proof 1 (via sequential compactness): (Theorem 4.4.1, Abbott) Let $K \in \mathcal{K}_{\mathbb{R}} = \mathcal{K}_{\mathbb{R}}^s$ and $f \in C(\mathbb{R})$, and let us show that $f(K) \in \mathcal{K}_{\mathbb{R}} = \mathcal{K}_{\mathbb{R}}^s$ by showing that every sequence $(y_n)_{n\in\mathbb{N}}$ in $f(K)$ has a convergent subsequence with limit in $f(K)$. Since each $y_n \in f(K)$, $\exists x_n \in K$ for which $y_n = f(x_n)$. Since K is (sequentially) compact, the sequence $(x_n)_{n\in\mathbb{N}}$ in K has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $x_{n_k} \to x \in K$. By the Convergence Criterion (Theorem 23, Lecture 10), we have $y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$, because $x \in K$. ■

Proof 2 (via compactness): Let $K \in \mathcal{K}_{\mathbb{R}}$ and $f \in C(\mathbb{R})$, and let us show that $f(K) \in \mathcal{K}_{\mathbb{R}}$ by showing that every open cover $\mathcal{U} = \{U_i \mid i \in I\}$ of $f(K)$ ($f(K) \subseteq \bigcup_{i\in I} U_i$) has a finite subcover. Since $f \in C(\mathbb{R})$, Theorem 27, Lecture 10, says each $f^{-1}(U_i) \in \mathcal{T}_{\mathbb{R}}$, so $f^{-1}(\mathcal{U}) \stackrel{\text{def}}{=} \{f^{-1}(U_i) \mid i \in I\}$ is an open cover of K (apply f^{-1} to both sides of $f(K) \subseteq \bigcup_{i\in I} U_i$, then $K \subseteq \bigcup_{i\in I} f^{-1}(U_i)$). But K is compact, so $f^{-1}(\mathcal{U})$ has a finite subcover $f^{-1}(\mathcal{V}) \stackrel{\text{def}}{=} \{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$ still covering $K \subseteq \bigcup_{i=1}^n f^{-1}(U_i)$. Consequently, $\mathcal{V} \stackrel{\text{def}}{=} \{U_1, \dots, U_n\}$ is a finite subcover of $f(K)$ (just apply f to both sides of $K \subseteq \bigcup_{i=1}^n f^{-1}(U_i)$). ■

Theorem 2 (Extreme Value Theorem (EVT)) *If $\varnothing \neq K \in \mathcal{K}_{\mathbb{R}}$ and $f \in C(\mathbb{R})$, then $\exists a, b \in K$ such that*

$$f(a) = \min f(K) \equiv \min_{x \in K} f(x)$$
$$f(b) = \max f(K) \equiv \max_{x \in K} f(x)$$

that is, f attains or hits its minimum and maximum y-values on K.

Proof: If $K \in \mathcal{K}_{\mathbb{R}}$ and $f \in C(\mathbb{R})$, then $f(K) \in \mathcal{K}_{\mathbb{R}}$ by the previous theorem, so since $\mathcal{K}_{\mathbb{R}} = \mathcal{K}_{\mathbb{R}}^{cb}$ (by Heine-Borel, Theorem 16, Lecture 08), we know that $f(K)$ is closed and bounded. Since $f(K)$ is nonempty (because K is), $\exists m \overset{\text{def}}{=} \inf f(K)$ and $\exists M \overset{\text{def}}{=} \sup f(K)$ by AoC. But these are in the closure $\overline{f(K)} = f(K)$ (since $f(K) \in \mathcal{C}_{\mathbb{R}}$, Exercises 40-41, Lecture 07), so $m, M \in f(K)$, meaning $\exists a, b \in K$ for which $f(a) = m = \min f(K)$ and $f(b) = M = \max f(K)$ (since $m = \inf f(K) = \min f(K)$ and $M = \sup f(K) = \max f(K)$ because $m, M \in f(K)$ (Proposition 22, Lecture 02)). ■

1.2 Compactness and Uniform Continuity

Example 3 Consider $f(x) = x^2$. By Corollary 26, Lecture 10, $f \in C(\mathbb{R})$. However, the continuity is *not uniform*, in the sense that, though $\forall a \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists \delta > 0$ satisfying $\big(|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon\big)$, the δ *depends on both* ε *and* x. To see this, recall that

$$|x^2 - a^2| = |x - a||x + a|$$

and we need to bound $|x + a|$. From $\delta \leq 1$ we derive

$$\begin{aligned} |x - a| \leq 1 &\iff -|a| - 1 \leq a - 1 < x < a + 1 \leq |a| + 1 \\ &\iff -2|a| - 1 \leq 2a - 1 < x + a < 2a + 1 \leq 2|a| + 1 \\ &\iff |x + a| < 2|a| + 1 \end{aligned}$$

(or alternatively, $|x + a| \leq |x| + |a| \leq (|a| + 1) + |a| = 2|a| + 1$, using only the first equivalence). Then we can say that

$$\forall a \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \delta \overset{\text{def}}{=} \min\left\{1, \frac{\varepsilon}{2|a| + 1}\right\} > 0,$$
$$\left(|x - a| < \delta \implies \left\{ \begin{aligned} |x^2 - a| &= |x - a||x + a| \\ &< \delta \cdot (2|a| + 1) \\ &\leq \frac{\varepsilon}{2|a| + 1} \cdot (2|a| + 1) \\ &= \varepsilon \end{aligned} \right\} \right)$$

The key observation: $\delta = \delta(\varepsilon, a)$! Larger a require smaller δ, so that no single delta works for all $a \in \mathbb{R}$. Any δ will *not be uniformly applicable* on $\mathbb{R}$. ■

Definition 4 Let $A \subseteq \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to be **uniformly continuous on** A if

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x, y \in A,\ \big(|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon\big)$$

Let us denote the **set of all uniformly continuous functions on** A by

$$C^u(A) \overset{\text{def}}{=} \textbf{all uniformly continuous functions on } A$$

Note that this is not the case for $f(x) = x^2$ on $A = \mathbb{R}$, because for a fixed $\delta > 0$, the difference $|x^2 - y^2| = |x - y||x + y|$ will be bigger for large x, y satysfying $|x - y| < \delta$ than for small x, y (since $|x + y|$ can be made large while keeping $|x - y| < \delta$).

Theorem 5 (Compactness and Uniform Continuity) *If* $K \in \mathcal{K}_{\mathbb{R}}$ *and* $f \in C(K)$, *then* $f \in C^u(K)$.

Proof: Let $\varepsilon > 0$ and use the continuity of f to find, for each $x \in K$, a $\delta_x > 0$ such that $|x - y| < \delta_x \implies |f(x) - f(y)| < \varepsilon/2$. Now, $\mathcal{U} \overset{\text{def}}{=} \{V_{\frac{\delta_x}{2}}(x) \mid x \in K\}$ covers K, but since K is compact, $\mathcal{U}$ has a finite subcover $\mathcal{V} = \{V_{\frac{\delta_{x_1}}{2}}(x_1),\ \ldots,\ V_{\frac{\delta_{x_n}}{2}}(x_n)\}$. Letting $\delta \overset{\text{def}}{=} \min\{\frac{\delta_{x_1}}{2}, \ldots, \frac{\delta_{x_n}}{2}\}$ we have for each $x \in K \subseteq \bigcup_{i=1}^n V_{\frac{\delta_{x_i}}{2}}(x_i)$ that $x \in V_{\frac{\delta_{x_i}}{2}}(x_i)$ for one of the i. Consequently, if $y \in K$ satisfies $|x - y| < \delta$, then

$$|x_i - y| \leq |x_i - x| + |x - y| < \frac{\delta_{x_i}}{2} + \delta \leq \delta_{x_i}$$

By our choice of δ_{x_i} we have $|f(y) - f(x_i)| < \frac{\varepsilon}{2}$ and $|f(x_i) - f(x)| < \frac{\varepsilon}{2}$, so

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \varepsilon$$

and $f \in C^u(K)$. ■

Theorem 6 *Let* $A \subseteq \mathbb{R}$. *Then* $f \notin C^u(A) \iff \exists \varepsilon_0 > 0$ *and* $\exists (a_n)_{n \in \mathbb{N}},\ (b_n)_{n \in \mathbb{N}} \in A^{\mathbb{N}}$ *satisfying* $|a_n - b_n| \to 0$ *but* $|f(a_n) - f(b_n)| \geq \varepsilon_0$ *for all* n.

Proof: Exercise! See Theorem 4.4.5, Abbott. ■

2 Continuous Functions on Connected Subsets of $\mathbb{R}$

2.1 Preservation of Connected and Intermediate Value Theorem (IVT)

Theorem 7 (Continuous Functions Preserve Connectedness) *If $A \subseteq \mathbb{R}$ is connected and $f \in C(A)$, then $f(A)$ is connected.*

Proof: Suppose $f(A) = B \cup C$ where $A, B \neq \varnothing$ and $A \cap B = \varnothing$. We will use Proposition 5, Lecture 9, to show that $B \cap L(C) \neq \varnothing$ or $L(B) \cap C \neq \varnothing$. Since $f(A) = B \cup C$, taking f^{-1} of both sides gives $A = f^{-1}(B) \cup f^{-1}(C)$, but A is connected, and $f^{-1}(B), f^{-1}(C) \neq \varnothing$ (because $B, C \neq \varnothing$) and $f^{-1}(B) \cap f^{-1}(C) = \varnothing$ (because $B \cap C = \varnothing$). Therefore, either $L(f^{-1}(B)) \cap f^{-1}(C) \neq \varnothing$ or $f^{-1}(B) \cap L(f^{-1}(C)) \neq \varnothing$. WOLOG, suppose the first case, and choose $x \in L(f^{-1}(B) \cap f^{-1}(C)$. Then $\exists (a_n)_{n\in\mathbb{N}}$ in $f^{-1}(B) - \{x\}$ with $a_n \to x \in f^{-1}(C)$, which shows that $\big(f(a_n)\big)_{n\in\mathbb{N}}$ lies in $B - \{f(x)\}$ and $f(a_n) \to f(x) \in C$, that is $f(x) \in L(B) \cap C$. ■

Theorem 8 (Intermediate Value Theorem (IVT)) *If f is continuous on a closed and bounded interval $[a, b]$ in $\mathbb{R}$, then every intermediate y-value N between $f(a)$ and $f(b)$ is attained, that is, $\exists c \in [a, b]$ satisfying $f(c) = N$.*

Proof: Exercise 4.5.1, Abbott. ■

Proposition 9 *Let $\varnothing \neq I \in \mathcal{I}$ be a nonempty interval and let $f \in C(I)$. Then f is bijective between I and $f(I)$ iff it is strictly monotonic on I.*

Proof:

(1) Suppose f is bijective on I. Let a and b be the endpoints of I, where $a < b$, so that $(a, b) \subseteq I$. Since f is bijective, for any $a < x_0 < y_0 < b$ we can't have $f(x_0) = f(y_0)$, so we must either have $f(x_0) < f(y_0)$ or $f(x_0) > f(y_0)$. Suppose $f(x_0) < f(y_0)$. We claim that for any $a < x < y < b$ we will have $f(x) < f(y)$ and hence that f is strictly increasing on I. For suppose $\exists a < x_0 < x < y_0 < b$ with $f(x_0) < f(y_0)$ yet with $f(x_0) > f(x)$. Then by EVT $\exists c \in (x_0, y_0)$ with $f(c) = \min f([x_0, y_0])$, while by IVT applied to f on $[x_0, c]$ and on $[c, y_0]$, we get $\forall y \in (f(c), f(x_0))$, $\exists x_1 \in (x_0, c)$, $\exists x_2 \in (c, y_0)$ with $f(x_1) = y = f(x_2)$. But $x_1 < x_2$, so this contradicts the bijectivity of f. We conclude that no such $x_0 < x < y_0$ exist, and that f must be strictly increasing on I. A similar argument, for the case $f(x_0) > f(y_0)$, shows that f must then be strictly decreasing on I.

(2) Now suppose f is strictly monotonic on $I \supseteq (a, b)$. If f is increasing, then $a < x < y < b \implies f(x) < f(y)$, so that f is injective, and by IVT we know that f is surjective: $\forall z \in \big(f(x),\ f(y)\big) = f\big((x, y)\big)$, $\exists c \in (x, y)$ with $f(c) = y$. A similar argument applies to f decreasing. If need be, we could include the endpoints a and b themselves, and so extend the argument to closed and half-open intervals. ■

Proposition 10 *Let $K \in \mathcal{K}_{\mathbb{R}}$ be compact. Then any bijective $f \in C(K)$ is in fact a* ***homeomorphism*** *(continuous bijection with continuous inverse) between K and $f(K)$.*

Proof: Since $f \in C(K)$, we know that $f^{-1}(\mathcal{T}_{f(K)}) \subseteq \mathcal{T}_K$ and $f^{-1}(\mathcal{C}_{f(K)}) \subseteq \mathcal{C}_K$. If f is additionally bijective, then to show that $f^{-1} \in C(f(K))$ we need to show that $f(\mathcal{T}_K) = (f^{-1})^{-1}(\mathcal{T}_K) \subseteq \mathcal{T}_{f(K)}$ or $f(\mathcal{C}_K) = (f^{-1})^{-1}(\mathcal{C}_K) \subseteq \mathcal{T}_{f(K)}$. Let $C \in \mathcal{C}_K$, then $C \subseteq K$ is compact (Proposition 15, Lecture 08), so $f(C) \subseteq f(K)$ is also compact (Theorem 1), and therefore closed by Heine-Borel (Theorem 16, Lecture 08). We conclude that $f(C) \in \mathcal{C}_{f(K)}$. ■

Theorem 11 *Let $\varnothing \neq I \in \mathcal{I}$ be a nonempty interval and let $f \in C(I)$.*

(1) *If f is strictly increasing on I, then f^{-1} is strictly increasing, too, and also continuous, $\boxed{f^{-1} \in C(f(I))}$. Thus, f is a* ***homeomorphism*** *from I to $f(I)$.*

(2) *If f is strictly decreasing in I, then f^{-1} is strictly decreasing, too, and also continuous, $\boxed{f^{-1} \in C(f(I))}$. Thus, f is a* ***homeomorphism*** *from I to $f(I)$.*

Proof: (Exercise 4.5.8, Abbott)

(1) Suppose first that $I = [a, b]$ is closed. Since $f \in C[a, b]$, EVT + IVT tell us $f([a, b]) = [m, M]$ is a closed and bounded interval, where $m = \min_{a \le x \le b} f(x)$ and $M = \max_{a \le x \le b} f(x)$. If f is increasing, then $m = f(a)$ and $M = f(b)$, so that $f([a, b]) = [f(a), f(b)]$, and f is a bijection between $[a, b]$ and $[f(a), f(b)]$. Since $[a, b]$ is compact (Heine-Borel, Theorem 16, Lecture 08), and since f is bijective, we have by the previous proposition that f is a homeomorphism between $[a, b]$ and $[f(a), f(b)]$, so that $f^{-1} \in C([f(a), f(b)])$. It is also obviously strictly increasing. A similar argument applies to f decreasing.

(2) If $I = (a, b)$ or $(a, b]$ or $[a, b)$, then the above statement could be gotten by restricting attention to closed and bounded subintervals $[x_0, y_0]$ of I. Clearly $f^{-1} \in C(f([x_0, y_0]))$, and so $f^{-1} \in C\big(\bigcup_{x_0, y_0 \in (a,b)} f([x_0, y_0])\big) = C(f(a, b))$. The limits $\lim_{x_0 \searrow a} f(x_0)$ or $\lim_{x_0 \nearrow b} f(x_0)$ may or may not be realized by f, and these are precisely the three options listed above corresponding to the inclusion or exclusion of an endpoint into/from I. ■