10: Continuous Functions on $\mathbb{R}$
Part I

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Contents

1 Why Do We Need to Define a Continuous Function?

We are finally in a position to develop the theory of the calculus. Having characterized the one truly essential property of $\mathbb{R}$, the Axiom of Completeness (AoC), and seen some of its equivalent formulations (NIP, MCT, BW, CC), we will now finish its topological characterization by addressing the **continuous functions on** $\mathbb{R}$. Without this notion the derivative cannot be properly defined, and the calculus cannot be solidly grounded.

1.1 Math as a Craft

Every student of calculus comes away knowing the **rules of differentiation** which break down into the *particular derivative rules* of known functions and the *generic rules* concerning *algebraic combinations* of differentiable functions:

(1) the **particular derivatives**, starting with the *power rule*,

$$(x^n)' = nx^{n-1}$$

the *exponential and log derivative rules*,

$$\begin{aligned} (e^x)' &= e^x & (\ln x)' &= \tfrac{1}{x} \\ (a^x)' &= a^x \ln a & (\log_a x)' &= \tfrac{1}{x \ln a} \end{aligned}$$

and the *trig derivative rules*,

$$\begin{aligned} (\sin x)' &= \cos x & (\cos x)' &= -\sin x \\ (\sec x)' &= \sec x \tan x & (\csc x)' &= -\csc x \cot x \\ (\tan x)' &= \sec^2 x & (\cot x)' &= -\csc^2 x \end{aligned}$$

(2) the **generic rules** for all differentiable functions f and g and all constants c, starting with the *sum/differentce* and *scalar multiplication rules*,

$$(f \pm g)'(x) = f'(x) \pm g'(x), \qquad (cf)'(c) = cf'(x),$$

the *product and quotient rules*,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x), \qquad \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

and, lastly, the *inverse function rule* (valid whenever the denominator $\neq 0$),

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$

Then, come the *Mean Value Theorem*, the *First and Second Derivative Tests*, and *L'Hôpital's Rule*, and before we know it we're on to Riemann integral.

The **definition of the derivative**,

$$f'(a) \stackrel{\text{def}}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

comes almost as an *annoyance*, a *technical afterthought* distracting us from the easy and rather mechanical application of the derivative rules, by which we can quickly finish homework problems and get back to our favorite activity, *free play*. Isn't the purpose of rules to free us from the burden of tedious checking of cases, after all?

We endure the definition as we must, and the mandatory list of tedious and artificial "counterexamples" (which seem to almost invalidate the rules), in order to get through the exam, following which we forget about them. We retain only the rules, validating in this way Father Guido Sarducci's pedagogical principle, engraved upon the entrance to his *Five Minute University*, that "One could learn in five minutes what the average college graduate remembers five years after graduation."

Figure 1: Father Guido's Five Minute University.

But this is like the California Gold Rush, when the makers of shovels and sieves made fortunes selling to amateur miners all too eager to rush out and start digging up the fabled gold. In our practically oriented culture we are encouraged to think of math as a mere tool, like a shovel, which we can just "apply" to dig out the gold of applied problems. Yet here, as in the "real world," it is in fact the tool-*makers* who come away with the gold. In a very real sense, the craft of tool-making is prior to the other crafts which depend on the tool.

The craft of tool-making has to be appreciated on its own merits. Mathematics produces conceptual tools, very refined and sharp, whose true value is readily appreciated by its users, the appliers of math. Why are these tools so sharp, what is their design? From this point of view, the idea that definitions and subtle counterexamples are tedious "work," like bureacratic paperwork that has to be filled out in order to get back to free play, is the thinking of a tool user, not a tool maker. *Our* craft is the proper construction of tools, and for this purpose definitions are essential. They *define* the problem which the subsequent theory sets out to resolve, *in precise terms*. The whole art consists in coming up with the right definitions, the right axioms, the right ordering of concepts in such a way that the deeper ideas flow elegantly and naturally out of the axioms, with utmost precision. A good theory, when parsed for its meaning, should render a systematic classification of its concepts, which have been clearly articulated in the definitions.

1.2 Classification of Functions

Let us keep this larger classification goal in mind. The *role of definitions* in such an endeavour is to articulate the *criteria for differentiation* between distinct conceptual types (e.g. continuous vs. discontinuous functions, differentiable vs non-differentiable functions, differentiable vs. continuous functions, etc.). The *job of the theorems and examples* is to *map out and refine the classification* of the different types (e.g. differentiable functions are continuous, but there are more continuous functions than just the differentiable (as shown by a counterexample)). Ideally, the classification would be exhaustive, capable of answering any question within its purview with perfect precision.

Observation 1 If we take the convergence issues implicit in the **definitions of functional limit**, **continuity** and the **derivative** seriously, we discern quickly enough the depths lurking beneath the tranquil surface. Let $f : A \to B$, where $A \subseteq \mathbb{R}$ is the domain, and $f(A) \subseteq B$ is the range (the codomain is B).

(1) How should we characterize the **functional limit** L **at** $x = a$,

$$\boxed{\lim_{x \to a} f(x) = L}$$

It seems somewhat of a topological requirement,

$$x \to a \in L(A) \quad \Longrightarrow \quad f(x) \to L \in L(f(A)) \cup I(f(A))$$

(2) What about **continuity at** $x = a$,

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

Now it appears that

$$x \to a \in A \quad \Longrightarrow \quad f(x) \to f(a) \in \underbrace{\big(f(A) \cap L(f(A))\big) \cup I(A)}_{= f(A)}$$

(3) How are we to understand the **derivative at** $x = a$,

$$\boxed{\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)}$$

It seems we are back to the functional limit, with $L = f'(a)$, but with f replaced by $g(x) \stackrel{\text{def}}{=} \frac{f(x) - f(a)}{x - a}$,

$$x \to a \in L(A) \quad \Longrightarrow \quad g(x) \to f'(a) \in L(g(A)) \cup I(g(A)) \qquad \blacksquare$$

While functional limits are a necessary part of defining continuity, the really critical thing for continuity is that $L = f(a)$, so that $a \in A$ and not just in $L(A)$. In practice, we use the functional limit to determine *dis*continuities (namely, those for which L exists but $L \neq f(a)$).

When we get to differentiability, however, we see an entirely different requirement on f. By our definitions, differentiability implies continuity, but the converse does not hold. This sets continuous functions apart from differentiable functions. Continuity is more basic, more primitive, and differentiability a sort of "extra" feature. Let us spell these observations out in terms of classification and the AoC.

Observation 2 One reason to make a distinction between *continuous* and *differentiable* functions on A concerns *classification*. The set of all differentiable functions on A, $\mathcal{D}(A)$, forms a strictly smaller subset of the set of all continuous functions on A, $C(A)$,

$$\boxed{\mathcal{D}(A) \subsetneq C(A)}$$

This says that analysis, which in its differential aspect studies subspaces of $\mathcal{D}(A)$, concerns itself here with a strictly smaller subcategory than the larger, and more basic, parent category of topology. ■

Observation 3 Another reason to make a distinction between *continuous* and *differentiable* functions on A concerns the Axiom of Completeness. *The Intermediate Value Theorem is logically equivalent to the AoC*, whereas differentiability is a much more stringent requirement. ■

2 Defining a Continuous Function

2.1 Adherent Points

Definition 4 Let A be a subset of $\mathbb{R}$. We say $x \in \mathbb{R}$ is an **adherent point** of A if

$$\boxed{\forall \varepsilon > 0,\ V_\varepsilon(x) \cap A \neq \varnothing}$$

We denote the **set of all adherent points of** A by

$$\boxed{\begin{aligned} \mathsf{A}(A) &\overset{\text{def}}{=} \textbf{all adherent points of } A \\ &= \{x \in \mathbb{R} \mid \forall \varepsilon > 0,\ V_\varepsilon(x) \cap A \neq \varnothing\} \end{aligned}}$$

Exercise 5 *Show that for any* $A \subseteq \mathbb{R}$ *we have* $\boxed{\mathsf{A}(A) = \overline{A} = L(A) \cup I(A)}$. ■

Remark 6 *This exercise shows that the condition on* $x \in L(A)$*, that* $\forall \varepsilon > 0$ *we have* $V_\varepsilon(x) \cap (A - \{x\}) \neq \varnothing$*, when weakened to* $V_\varepsilon(x) \cap A \neq \varnothing$ *forces isolated points to be adherent points,* $I(A) \subseteq \mathsf{A}(A)$*, and in fact* $\mathsf{A}(A) - L(A) = I(A)$. ■

2.2 Functional Limits

Let $f : A \to B$, where $A, B \subseteq \mathbb{R}$.

Definition 7 (Functional Limit) We say f has a **functional limit** of L at a limit point $\boxed{x = a \in L(A)}$, and write

$$\boxed{\lim_{x \to a} f(x) = L}$$

if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \big(0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon\big)}$$ ■

Remark 8 *In terms of neighborhoods, this means*

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \Big(x \in \underbrace{V_\delta(a) \cap (A - \{a\})}_{\neq \varnothing,\ \text{bec } a \in L(A)} \implies \underbrace{f(x) \in \overbrace{V_\varepsilon(L) \cap f(A)}^{\neq \varnothing\ \text{bec } f(x) \in}}_{L \in \mathsf{A}(f(A))}\Big)}$$

Topologically, this forces L *to be an adherent point of* $f(A)$*,*

$$\boxed{L \in \mathsf{A}\big(f(A)\big) = L(f(A)) \cup I(f(A))}$$

It may or may not be the case that $L \in f(A)$, but if it is, we do not remove it (so as to get $V_\varepsilon(a) \cap (f(A) - \{L\})$), because we want to allow constant functions $f(x) \equiv L$ to have a limit. In their case, $f(A) = \{L\}$ is isolated, so it is not a limit point. Indeed, in this case $f(A) - \{L\} = \{L\} - \{L\} = \varnothing$. ■

Remark 9 *Or yet another way to say it: $\forall \varepsilon > 0$, $\exists \delta > 0$, so that the graph of f over $V_\delta(a) \cap (A - \{a\})$ lies entirely in the divided rectangle*

$$\operatorname{graph} f \subseteq \underbrace{\big((a-\delta,\ a) \cup (a,\ a+\delta)\big)}_{= V_\delta(a) - \{a\}} \times \underbrace{(L-\varepsilon,\ L+\varepsilon)}_{= V_\varepsilon(L)}$$

a subset of $V_\delta(a) \times V_\varepsilon(L)$ missing $\{a\} \times V_\varepsilon(L)$. ■

Example 10 Consider the example of $\dfrac{\sin x}{x}$ on $A = \mathbb{R} - \{0\}$.

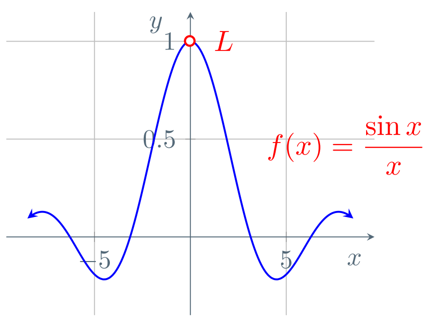
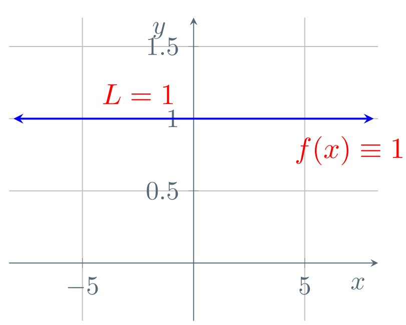


Figure 2: $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Here, $\boxed{A = (-\infty, 0) \cup (0, \infty)}$, $\boxed{a = 0 \in L(A) - A}$ and $\boxed{L = 1 \in L(f(A)) - f(A)}$. ■

Example 11 Consider the example of the constant function $f(x) \equiv 1$.



Here, $\boxed{A = \mathbb{R}}$, any $\boxed{a \in A}$ will do, and $\boxed{L = 1 \in f(A) = \{1\}}$. ■

Theorem 12 (Convergence Criterion for the Limit of a Function) *Let $A \subseteq \mathbb{R}$, and let $f : A \to \mathbb{R}$ be any function. If $a \in L(A)$, then*

$$\lim_{x \to a} f(x) = L \iff \lim_{n \to \infty} f(a_n) = L$$

for all sequences $(a_n)_{n\in\mathbb{N}}$ in $A - \{a\}$ with $a_n \to a$.

Proof:

(1) If $\lim_{x \to a} f(x) = L$, then $\forall \epsilon > 0,\ \exists \delta > 0$, such that

$$f\big(V_\delta(a) \cap (A - \{a\})\big) \subseteq V_\varepsilon(L) \cap f(A)$$

Consequently, if $(a_n)_{n\in\mathbb{N}}$ is a sequence in $A-\{a\}$ converging to a, then $\exists N \in \mathbb{N}$ such that $n \geq N \implies a_n \in V_\delta(a) \cap (A - \{a\})$. Our assumptions about f then force all $f(a_n)$ into $V_\varepsilon(L) \cap f(A)$, meaning $|f(a_n) - L| < \varepsilon$, whence $\lim_{n\to\infty} f(a_n) = L$.

(2) Contrapositive: If $\lim_{x\to a} f(x) \neq L$, then $\exists \varepsilon > 0$, $\forall \delta > 0$ such that

$$f\big(V_\delta(a) \cap (A - \{a\})\big) \not\subseteq V_\varepsilon(L) \cap f(A)$$

suggesting that, for this $\varepsilon > 0$, letting $\delta_n = \frac{1}{n} \searrow 0$ (and so making the original sequence a_n converge to a), we have

$$\forall n \in \mathbb{N},\ \exists f(a_n) \in f\big(V_{\frac{1}{n}}(a) \cap (A - \{a\})\big) - V_\varepsilon(L)$$
$$\big(\text{that is } |f(a_n) - L| \geq \varepsilon \text{ instead of } < \varepsilon\big)$$

that is, $f(a_n) \not\to L$. ■

Theorem 13 (Alegebraic Limit Laws for Functional Limits) *Let $A \subseteq \mathbb{R}$ and $a \in L(A)$, and let $f, g : A \to \mathbb{R}$ be any functions. If*

$$\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M$$

exist, then the functions cf, $f \pm g$, fg and f/g (if $g \neq 0$) also have limits, and

$$\begin{aligned}
\lim_{x\to a} cf(x) &= cL && \text{(scalar multiple law)}\\
\lim_{x\to a} f(x) \pm g(x) &= L \pm M && \text{(sum/difference law)}\\
\lim_{x\to a} f(x)g(x) &= LM && \text{(product law)}\\
\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{L}{M} && \text{(quotient law)}
\end{aligned}$$

Proof: Since $\lim_{x\to a} f(x) = L \iff \lim_{n\to\infty} f(a_n) = L$ for all sequences $a_n \to a$ in $A - \{a\}$, we can apply the Algebraic Limit Laws for Sequences. ■

Remark 14 *The collection of all functions $f : A \to \mathbb{R}$ possessing a limit L at $a \in L(A)$ forms a real vector space, and in fact, because it also possesses a product, forms an $\mathbb{R}$-algebra. This is not a particularly interesting $\mathbb{R}$-algebra, but it will become so when we replace L with $f(a)$, i.e. when we go over to continuous functions.* ■

Theorem 15 (Order Limit Laws for Functions) *Let $A \subseteq \mathbb{R}$ and $a \in L(A)$, and let $f, g : A \to \mathbb{R}$ be any functions. If*

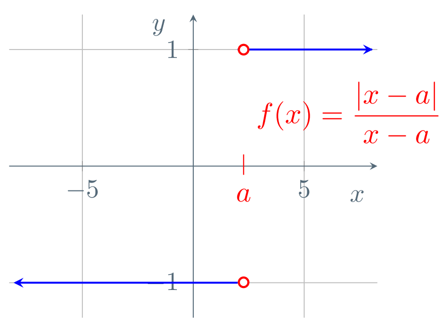
$$\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M$$

exist, then

$$\begin{aligned}(\forall x \in A,\ f(x) \geq 0) &\implies L \geq 0 \\ (\forall x \in A,\ f(x) \leq g(x)) &\implies L \leq M\end{aligned}$$

Proof: Since $\lim_{x\to a} f(x) = L \iff \lim_{n\to\infty} f(a_n) = L$ for all sequences $a_n \to a$ in $A - \{a\}$, we can apply the Order Limit Laws for Sequences. ■

Example 16 Consider the function $f(x) = \dfrac{|x-a|}{x-a} = \begin{cases} 1, & \text{if } x \geq a \\ -1, & \text{if } x \leq a \end{cases}$



Here, $\boxed{A = (-\infty, a) \cup (a, \infty)}$, $\boxed{a \in L(A) - A}$ and $\boxed{\lim_{x\to a} f(x) \text{ DNE}}$, because if we take $\varepsilon = 1$, then for all $\delta > 0$ there exists $a < x \in V_\delta(a) \cap (A - \{a\})$ for which $|f(x) - (-1)| = |1 + 1| = 2 \geq 1 = \varepsilon$ and there exists $a > x \in V_\varepsilon(a) \cap (A - \{a\})$ for which $|f(x) - 1| = |-1 - 1| = 2 \geq 1 = \varepsilon$. That is, $L \neq \pm 1$, and indeed, no other choices for L suffice either, by analogous reasoning. We conclude that L DNE. ■

Definition 17 (Left and Right Limits) We say f has a **left (functional) limit** of L^- at a limit point $a \in L(A)$, and write

$$\boxed{\lim_{x \to a^-} f(x) = L^-}$$

if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \left(0 < a - x < \delta \implies |f(x) - L^-| < \varepsilon\right)}$$

$(0 < a - x < \delta \iff a - \delta < x < a)$ We say f has a **right (functional) limit** of L^+ at a limit point $a \in L(A)$, and write

$$\boxed{\lim_{x \to a^+} f(x) = L^+}$$

if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \left(0 < x - a < \delta \implies |f(x) - L^+| < \varepsilon\right)}$$

$(0 < x - a < \delta \iff a < x < a + \delta)$. ■

Proposition 18 (Two-Sided Limits) *Let $A \subseteq \mathbb{R}$, $a \in L(A)$, and $f : A \to \mathbb{R}$. Then,*

$$\lim_{x \to a} f(x) = L \textit{ exists} \iff L^{\pm} = \lim_{x \to a^{\pm}} f(x) \textit{ exist and } L^+ = L^-$$

$$\textit{(in which case } L = L^{\pm}\textit{)}$$

Proof: (Exercise 4.2.10, Abbott)

(1) If L exists, then $L^{\pm} = L$ exist too because $\forall \varepsilon > 0$, $\exists \delta > 0$ with $0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$. Now, the δ-inequality expands into a double inequality,

$$0 < |x - a| < \delta \iff a - \delta < x < a \quad \text{or} \quad a < x < a + \delta$$

from which we see that $x \to a^-$ uses $a - \delta < x < a$ and $x \to a^+$ uses $a < x < a + \delta$ to show that $|f(x) - L| < \varepsilon$, that is $L = \lim_{x \to a^{\pm}} f(x) = L^{\pm}$.

(2) If $\lim_{x \to a^{\pm}} f(x) = L^+ = L^- = \lim_{x \to a^-} f(x)$, then let $L \overset{\text{def}}{=} L^{\pm}$ and reverse the logic above: $x \to a^-$ uses $a - \delta < x < a$ to show $|f(x) - L^-| < \varepsilon$ there, and $x \to a^+$ uses $a < x < a + \delta$ to show $|f(x) - L^+| < \varepsilon$ there, but the double inequality folds into one,

$$a - \delta < x < a \quad \text{or} \quad a < x < a + \delta \iff 0 < |x - a| < \delta$$

which shows $f(x) \to L \overset{\text{def}}{=} L^{\pm}$ in the functional limit sense. ■

2.3 Continuous Functions

Definition 19 Let $A \subseteq \mathbb{R}$. We call $B \subseteq A$ **(relatively) open in** A if $B = O \cap A$ for some $O \in \mathcal{T}_{\mathbb{R}}$, and we call $B \subseteq A$ **(relatively) closed in** A if $B = C \cap A$ for some $C \in \mathcal{C}_{\mathbb{R}}$. Let us denote the set of (relatively) open subsets of A by $\boxed{\mathcal{T}_A}$ and the set of (relatively) closed subsets of A by $\boxed{\mathcal{C}_A}$. ■

Let $f : A \to B$, where $A, B \subseteq \mathbb{R}$.

Definition 20 (Continuity) We say that f is **continuous at a point** $a \in A$, and write

$$\boxed{\lim_{x \to a} f(x) = f(a)}$$

if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \big(|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon\big)}$$

We say f is **continuous on a set** A if it is continuous at all $a \in A$. Let

$$\boxed{C(A) \overset{\text{def}}{=} \{f : A \to \mathbb{R} \mid f \text{ is continuous on } A\}}$$

denote the **set of all continuous functions on** A. ■

Remark 21 *In terms of neighborhoods, this means*

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\ \Big(\underbrace{x \in \overbrace{V_\delta(a) \cap A}^{\in \mathcal{T}_A}}_{x,\, a \in A} \implies \underbrace{f(x) \in \overbrace{V_\varepsilon\big(f(a)\big) \cap f(A)}^{\in \mathcal{T}_{f(A)}}}_{f(x),\, f(a) \in f(A)}\Big)}$$

or equivalently

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0,\quad f\big(V_\delta(a) \cap A\big) \subseteq V_\varepsilon\big(f(a)\big) \cap f(A)}$$

It is now the case that $a \in \mathsf{A}(A)$, but specifically in the part $\mathsf{A}(A) \cap A = A$. Note that we could have $a \in I(A)$, in which case we have defined continuity at an isolated point. Aditionally, we now not only have $f(a) \in \mathsf{A}(f(A))$, but specifically $f(a) \in \mathsf{A}(f(A)) \cap f(A) = f(A)$. ■

Remark 22 *We break down the defining condition of continuity into three parts:*

$$\boxed{\lim_{x \to a} f(x) = f(a) \iff \left\{\begin{array}{ll} (1) & \exists L = \lim_{x \to a} f(x) \\ (2) & a \in A \quad \textit{(so that } f(a) \textit{ is defined)} \\ (3) & L = f(a) \end{array}\right\}}$$ ■

Immediately from the functional limit convergence criterion and limit laws we have the following analogues for continuous functions:

Theorem 23 (Convergence Criterion for Continuous Functions)
Let $A \subseteq \mathbb{R}$, and let $f : A \to \mathbb{R}$. For any $a \in A$,

$$\lim_{x\to a} f(x) = f(a) \quad \Longleftrightarrow \quad \lim_{n\to\infty} f(a_n) = f(a) = f(\lim_{n\to\infty} a_n)$$

for all sequences $(a_n)_{n\in\mathbb{N}}$ in A with $a_n \to a$.

Proof: See Theorem 12 above. ■

Theorem 24 (Algebraic Limit Laws for Continuous Functions)
If $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at $a \in A$, then the functions cf, $f \pm g$, fg and f/g (if $g \neq 0$) are also continuous at a, and

(1) $\lim_{x\to a} cf(x) \;=\; cf(a) \;=\; c\lim_{x\to a} f(x)$

(2) $\lim_{x\to a} f(x) \pm g(x) \;=\; f(a) \pm g(a) \;=\; \big(\lim_{x\to a} f(x)\big) \pm \big(\lim_{x\to a} g(x)\big)$

(3) $\lim_{x\to a} f(x)g(x) \;=\; f(a)g(a) \;=\; \big(\lim_{x\to a} f(x)\big)\big(\lim_{x\to a} g(x)\big)$

(4) $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} = \frac{\big(\lim_{x\to a} f(x)\big)}{\big(\lim_{x\to a} g(x)\big)} \qquad (g \neq 0)$

We conlcude that $\boxed{C(A) \text{ forms a real vector space, in fact an } \mathbb{R}\text{-algebra}}$.

Proof: See Theorem 13 above. ■

Notation 25 *Denote the* ***set of all polynomials with real coefficients*** *by*

$$\boxed{\mathbb{R}[x] \stackrel{\text{def}}{=} \left\{ p(x) = \sum_{k=0}^{n} a_k x^k \;\middle|\; a_k \in \mathbb{R},\; n \in \mathbb{N} \right\}}$$

and the ***set of all rational functions*** *by*

$$\boxed{\mathbb{R}(x) \stackrel{\text{def}}{=} \left\{ \frac{p(x)}{q(x)} \;\middle|\; p, q \in \mathbb{R}[x] \right\}}$$ ■

Corollary 26 *All polynomials are continuous on all of* $\mathbb{R}$*,*

$$\mathbb{R}[x] \subseteq C(\mathbb{R})$$

and all rational functions $p(x)/q(x) \in \mathbb{R}(x)$ *are continuous wherever* $q(x) \neq 0$*.*

Proof: (Example 4.3.5, Abbott) First of all, note that $f(x) = x \in C(\mathbb{R})$ because $\forall a \in \mathbb{R},\ \forall \varepsilon > 0,\ \exists \delta \overset{\text{def}}{=} \varepsilon$ such that $0 < |x-a| < \delta$ implies $|f(x) - f(a)| = |x-a| < \delta = \varepsilon$. As a result of the product law (3) above, we have $x^2 \in C(\mathbb{R})$, and by induction $x^k \in C(\mathbb{R})$. By the scalar multiple law (1) we have that $a_k x^k \in C(\mathbb{R})$ for all $a_k \in \mathbb{R}$, while by induction on the sum law (2) we have any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k \in C(\mathbb{R})$. Finally, apply law (4) for rational functions. ■

Theorem 27 (Open Set Characterization of Continuity) *For any subsets* $A, B \subseteq \mathbb{R}$ *and any* $f : A \to B$ *we have*

$$f \in C(A) \iff \Big(O \in \mathcal{T}_B \implies f^{-1}(O) \in \mathcal{T}_A\Big)$$

Proof:

(1) Let $f \in C(A)$ and let $O \in \mathcal{T}_B$. To show $f^{-1}(O) \in \mathcal{T}_A$, we must show that every point $a \in f^{-1}(O)$ is interior to $f^{-1}(O)$. Since f is continuous at a, we have
$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \big(|x-a| < \delta \implies |f(x) - f(a)| < \varepsilon\big)$$
or equivalently
$$\forall \varepsilon > 0,\ \exists \delta > 0,\ f\big(V_\delta(a) \cap A\big) \subseteq V_\varepsilon\big(f(a)\big) \cap f(A)$$
But O is open in B and $f(a) \in O$ ($O = U \cap B$ for some $U \in \mathcal{T}_{\mathbb{R}}$), so $\exists \varepsilon > 0$ such that $V_\varepsilon\big(f(a)\big) \cap B \subseteq O = U \cap B$ ($V_\varepsilon\big(f(a)\big) \subseteq U$). Using *this* ε, the required δ then gives $f(V_\delta(a) \cap A) \subseteq V_\varepsilon\big(f(a)\big) \cap B \subseteq O$, which shows (by taking preimages f^{-1} of both sides) that $V_\delta(a) \cap A \subseteq f^{-1}(O)$, so $f^{-1}(O) \in \mathcal{T}_A$.

(2) Conversely, suppose $\big(O \in \mathcal{T}_B \implies f^{-1}(O) \in \mathcal{T}_A\big)$, and let us show that $f \in C(A)$. Let $a \in A$ and choose $\varepsilon > 0$. Since $V_\varepsilon\big(f(a)\big) \cap B \in \mathcal{T}_B$, we know immediately that $f^{-1}\big(V_\varepsilon\big(f(a)\big) \cap B\big) \cap A \in \mathcal{T}_A$. Since a lies in this open set, it must be interior, so $\exists \delta > 0$ such that $V_\delta(a) \cap A \subseteq f^{-1}\big(V_\varepsilon\big(f(a)\big) \cap B\big) \cap A$, from which it follows directly that $f(V_\delta(a) \cap A) \subseteq V_\varepsilon\big(f(a)\big) \cap f(A)$ ($f(A) \subseteq B$), making f continuous at a. ■

Corollary 28 (Closed Set Characterization of Continuity) *For any subsets $A, B \subseteq \mathbb{R}$ and any $f : A \to B$ we have*

$$f \in C(A) \iff \Big(C \in \mathcal{C}_B \implies f^{-1}(C) \in \mathcal{C}_A\Big)$$

Proof: This follows immediately from the fact that $C \in \mathcal{C}_\mathbb{R} \iff C^c \in \mathcal{T}_\mathbb{R}$. Intersecting C with B gives $C \cap B \in \mathcal{C}_B \iff C^c \cap B \in \mathcal{T}_B$. Finally, observe that $f^{-1}(C^c) \cap A = (f^{-1}(C))^c \cap A \in \mathcal{T}_A \iff f^{-1}(C) \cap A \in \mathcal{C}_A$. Combine these facts and the previous theorem. ■

Theorem 29 (Compositions of Continuous Functions are Continuous) *Let $g : A \to B$ and $f : B \to C$ be functions, where $A, B, C \subseteq \mathbb{R}$. If $f \in C(B)$ and $g \in C(A)$, then $f \circ g \in C(A)$.*

Proof 1: If $O \in \mathcal{T}_C$, then by Theorem 27 $f^{-1}(O) \in \mathcal{T}_B$ and $g^{-1}(f^{-1}(O)) = (f \circ g)^{-1}(O) \in \mathcal{T}_A$, so again by Theorem 27 $f \circ g \in C(A)$. ■

Proof 2: If $a \in A$, then because $g \in C(A)$ we have $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x - a| < \delta \implies |g(x) - g(a)| < \varepsilon$. But since $f \in C(B)$ we have for all $\eta > 0$, $\exists \varepsilon > 0$ with $|g(x) - g(a)| < \varepsilon \implies |f(g(x)) - f(g(a))| < \eta$, so that $f \circ g$ is continuous at a. But this is true for all $a \in A$, so $f \circ g \in C(A)$. ■

Proof 3: By Theorem 23 f and g are convergence preserving, so for all $a \in A$ and all sequences $(a_n)_{n \in \mathbb{N}}$ in A converging to a we have

$$\begin{aligned} \lim_{n\to\infty} a_n = a \quad &\implies \quad \lim_{n\to\infty} g(a_n) = g(a) \\ &\implies \quad \lim_{n\to\infty} f(g(a_n)) = f(g(a)) \end{aligned}$$

and so $f \circ g$ is convergence preserving, which means that it is continuous at $a \in A$. But this is true for all $a \in A$, so $f \circ g \in C(A)$. ■

3 Examples

Example 30 Show that $\boxed{\lim_{x\to 3} 3x - 5 = 4}$ using the definition of a limit.

Proof:

(1) Scratch work:. Let $\varepsilon > 0$ and see how to find a $\delta > 0$ so that $0 < |x-3| < \delta$ forces $|f(x) - 4| < \varepsilon$: well,

$$|f(x) - 4| = |(3x-5) - 4| = |3x - 9| = 3|x-3|$$

It seems that $\delta \overset{\text{def}}{=} \frac{\varepsilon}{3} > 0$ will do the job.

(2) Proof:

$$\forall \varepsilon > 0,\ \exists \delta \overset{\text{def}}{=} \min\left\{1,\ \frac{\varepsilon}{3}\right\},$$

$$\left(|x-3| < \delta \implies \left\{ \begin{array}{rcl} |(3x-5)-4| & = & 3|x-3| \\ & < & 3\delta \\ & = & 3 \cdot \dfrac{\varepsilon}{3} \\ & = & \varepsilon \end{array} \right\} \right)$$

Example 31 Show that $\boxed{\lim_{x\to -3} x^2 = 9}$ using the definition of a limit.

Proof:

(1) Scratch work: Let $\varepsilon > 0$ and see how to find a $\delta > 0$ so that $0 < |x - (-3)| = |x+3| < \delta$ forces $|x^2 - 9| < \varepsilon$: well,

$$|x^2 - 9| = |x-3||x+3|$$

so knowing that $|x+3| < \delta$, we would like to know if we can bound $|x-3|$ by M, for then we could let $\delta = \frac{\varepsilon}{M}$. Let's try to bound $|x-3|$. Suppose $0 < \delta \leq 1$, to start, for then

$$\begin{array}{rcl} |x+3| < \delta \leq 1 & \iff & -1\,(-6) < x+3\,(-6) < 1\,(-6) \\ & \iff & -7 < x-3 < -5 < 7 \\ & \iff & |x-3| < 7 \end{array}$$

So, let $M = 7$ and $\delta = \frac{\varepsilon}{7}$.

(2) Proof: Since $|x+3| < \delta \leq 1 \iff |x-3| < 7$, we have

$$\forall \varepsilon > 0,\ \exists \delta \overset{\text{def}}{=} \min\left\{1,\ \frac{\varepsilon}{7}\right\},$$

$$\left(|x-3| < \delta \implies \left\{ \begin{array}{rcl} |x^2-9| & = & |x-3||x+3| \\ & < & 7\delta \\ & = & 7 \cdot \dfrac{\varepsilon}{7} \\ & = & \varepsilon \end{array} \right\} \right)$$

Example 32 Show that $\boxed{\lim_{x \to 25} \sqrt{x} = 5}$ using the definition of a limit.

Proof:

(1) Scratch work: Let $\varepsilon > 0$ and see how to find a $\delta > 0$ so that $0 < |x - 25| < \delta$ forces $|\sqrt{x} - 5| < \varepsilon$: well,

$$|x - 25| = |\sqrt{x} - 5||\sqrt{x} + 5| \iff \frac{|x - 25|}{|\sqrt{x} + 5|} = |\sqrt{x} - 5|$$

so if we could bound $|\sqrt{x} + 5|$ from below (away from 0), then we will have bound $\frac{1}{|\sqrt{x}+5|}$ above. Let's try to do this. Suppose $0 < \delta \leq 1$ to start, for then

$$\begin{aligned} |x - 25| < \delta \leq 1 &\iff -1 < x - 25 < 1 \\ &\iff 24 < x < 26 \\ &\iff 2\sqrt{6} + 5 < \sqrt{x} + 5 < \sqrt{26} + 5 \end{aligned}$$

Therefore, $2\sqrt{6} + 5 < |\sqrt{x} + 5|$, so $\frac{1}{|\sqrt{x}+5|} < \frac{1}{2\sqrt{6}+5}$.

(2) Proof: Since $|x - 25| < 1 \iff \frac{1}{|2\sqrt{6}+5|} < \frac{1}{2\sqrt{6}+5}$, we have

$$\forall \varepsilon > 0,\ \exists \delta \overset{\text{def}}{=} \min\Big\{ 1,\ \varepsilon(2\sqrt{6} + 5) \Big\},$$

$$\left(|x - 25| < \delta \implies \left\{ \begin{aligned} |\sqrt{x} - 5| &= \frac{|x - 25|}{|\sqrt{x} + 5|} \\ &< \delta \cdot \frac{1}{2\sqrt{6} + 5} \\ &= \varepsilon(2\sqrt{6} + 5) \cdot \frac{1}{2\sqrt{6} + 5} \\ &= \varepsilon \end{aligned} \right\} \right)$$

4 Exercises

Exercise 33 *Find functions $f, g : \mathbb{R} \to \mathbb{R}$ for which $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c$ but for which $\lim_{x\to a} g(f(x)) \neq c$.* ■

Exercise 34 *If $f \in C(\mathbb{R})$, show that $|f| \in C(\mathbb{R})$. Show by an example that the converse is false.* ■

Exercise 35

(a) *Fix $a \in \mathbb{R}$ and define $f(x) \stackrel{\text{def}}{=} |x - a|$. Show that $f \in C(\mathbb{R})$.*

(b) ***(Exercise 4.3.12, Abbott)*** *Let $C \in \mathcal{C}_{\mathbb{R}}$ be closed, and define*

$$f(x) \stackrel{\text{def}}{=} \inf\{\, |x-a| \;\big|\; a \in C\}$$

Show that $f \in C(\mathbb{R})$ and that $f \neq 0$ on C. ■

Exercise 36 ***(Exercise 4.3.10, Abbott)***

(a) *Show that for all $a, b \in \mathbb{R}$ we have*

$$\max\{a,b\} = \frac{1}{2}\big((a+b) + |a-b|\big) \quad \textit{and} \quad \min\{a,b\} = \frac{1}{2}\big((a+b) - |a-b|\big)$$

(b) *If $f, g \in C(A)$, show that, for $h(x) \stackrel{\text{def}}{=} \max\{f(x), g(x)\}$ and $k(x) \stackrel{\text{def}}{=} \min\{f(x), g(x)\}$ we have*

$$h \stackrel{\text{def}}{=} \max\{f, g\} \in C(A)$$
$$k \stackrel{\text{def}}{=} \min\{f, g\} \in C(A)$$

(c) *Using induction on part (b) we can show that if $f_1, \ldots, f_n \in C(A)$, then*

$$h \stackrel{\text{def}}{=} \max\{f_1, \ldots, f_n\} \in C(A)$$
$$k \stackrel{\text{def}}{=} \min\{f_1, \ldots, f_n\} \in C(A)$$

too. To see that this may fail in the infinite case, consider the infinite sequence of functions given by

$$f_n(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } |x| \geq \frac{1}{n} \\ n|x|, & \text{if } |x| < \frac{1}{n} \end{cases}$$

Show that each $f_n \in C(\mathbb{R})$, but that

$$h \stackrel{\text{def}}{=} \sup\{f_n \mid n \in \mathbb{N}\} \notin C(\mathbb{R})$$ ■